Ministry of Education and Science of Ukraine
National Mining University of Ukraine

Theoretical Mechanics

Dynamics

Summary of Lectures

Dnepropetrovsk
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Упорядник О.М.Долгов, канд. техн. наук, доц.

Відповідальний за випуск зав. кафедри будівельної та теоретичної механіки С.Є.Блохін, д-р техн. наук, проф.

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1. INTRODUCTION TO DYNAMICS. LAWS OF DYNAMICS

1.1. Basic Concepts and Definitions. Dynamics is that section of mechanics, which treats of the laws of motion of material bodies subjected to the action of forces.

The motion of bodies from a purely geometrical point of view was discussed in kinematics. Unlike kinematics, in dynamics the motion of bodies is investigated in connection with the acting forces and the inertia of the material bodies themselves.

The concept of force as a quantity characterizing the measure of mechanical interaction of material bodies was introduced in the course of statics. But in statics we treated all forces as constant, without considering the possibility of their changing with time. In real systems, though, alongside of constant forces (gravity can generally be regarded as an example of a constant force) a body is often subjected to the action of variable forces whose magnitudes and directions change when the body moves. Variable forces may be both applied (active) forces and the reactions of constraints.

Experience shows that variable forces may depend in some specific ways on time, on the position of a body, or on its velocity (examples of dependence on time are furnished by the tractive force of an electric locomotive whose rheostat is gradually switched on or off, or the force causing the vibration of a foundation of a motor with a poorly centered shaft; the Newtonian force of gravitation or the elastic force of a spring depend on the position of a body; the resistance experienced by a body moving through air or water depends on the velocity. In dynamics we shall deal with such forces alongside of constant forces. The laws for the composition and resolution of variable forces are the same as for constant forces.

The concept of inertia of bodies arises when we compare the results of the action of an identical force on different material bodies. Experience shows that if the same force is applied to two different bodies initially at rest and free from any other actions, in the most general case the bodies will travel different distances and acquire different velocities in the same interval of time.

Inertia is the property of material bodies to resist a change in their velocity under the action of applied forces. If, for example, the velocity of one body changes slower than that of another body subjected to the same force, the former is said to have greater inertia, and vice-versa. The inertia of any body depends on the amount of matter it contains.

The quantitative measure of the inertia of body, which depends on the quantity of matter in the body, is called the mass of that body. In mechanics mass m is treated as a scalar quantity which is positive and constant for every body. The measurement of mass will be discussed in the following article.

In the most general case the motion of a body depends not only on its aggregate mass and the applied forces, the nature of motion may also depend on the dimensions of the body and the mutual position of its particles (i.e., on the distribution of its mass).

In the initial course of dynamics, in order to neglect the influence of the dimensions and the distribution of the mass of a body, the concept of a material point, or particle, is introduced.

A particle is a material body (a body possessing mass) the size of which can be neglected in investigating its motion.

Actually any body can be treated as a particle when the distances traveled by its points are very great as compared with the size of the body itself. Furthermore, as will be shown in the dynamics of systems, a body in translator motion can always be considered as a particle of mass equal to the mass of the whole body.

Finally, the parts into which we shall mentally divide bodies in analyzing any of their dynamic characteristics can also be treated as material points.

Obviously, the investigation of the motion of a single particle should precede the investigation of systems of particles, and in particular of rigid bodies. Accordingly, the course of dynamics is conventionally subdivided into particle dynamics and the dynamics of systems of particles.

1.2. The Laws of Dynamics. The study of dynamics is based on a number of laws generalizing the results of a wide range of experiments and observations of the motions of bodies—laws that have been verified in the long course of human history.

The First Law (the Inertia Law*): a particle free from any external influences continues in its state of rest, or of uniform rectilinear motion, except in so far as it is compelled to change that state by impressed forces.* The motion of a body not subjected to any force is called motion under no forces, or inertial motion.

The inertia law states one of the basic properties of matter: that of being always in motion. It establishes the equivalence, for material bodies, of the states of rest and of motion under no forces.

A frame of reference for which the inertia law is valid is called an inertial system (or, conventionally, a fixed system). Experience shows that, for our solar system, an inertial frame of reference has its origin in the center of the sun and its axes pointed towards the so-called "fixed" stars. In solving most engineering problems a sufficient degree of accuracy is obtained by assuming any frame of reference connected with the earth to be an inertial system.

The Second Law (the Fundamental Law of Dynamics) establishes the mode in which the velocity of a particle changes under the action of a force. *It states: the product of the mass of a particle and the acceleration imparted to it by a force is proportional to the acting force; the acceleration takes place in the direction of the force.* Mathematically this law is expressed by the vector equation:

$$m\vec{w} = \vec{F} \ . \tag{1.1}$$

The second law of dynamics, like the first, is valid only for an inertial system. It can be immediately seen from the law that the measure of the inertia of a particle is its mass, since two different particles subjected to the action of the same force receive the same acceleration only if their masses are equal; if their masses are different, the particle with the larger mass (i.e., the more inert one) will receive a smaller acceleration, and vice versa.

A set of forces acting on a particle can, as we know, be replaced by a single resultant $\vec{R}$ equal to the geometrical sum of those forces. In this case the equation expressing the fundamental law of dynamics acquires the form:

$$m\vec{w} = \vec{R} \ or \ m\vec{w} = \sum \vec{F}_k \ . \tag{1.2}$$

Measure of mass. Eq. (1) makes it possible to determine the mass of a body if its acceleration in translator motion and the acting force are known. It has been established experimentally that under the action of the force of gravitation $\vec{P}$ all bodies falling to the earth (from a small height and in vacuum) possess the same acceleration g, this is known as the acceleration of gravity or of free fall. Applying Eq. (1.2) to this motion, we obtain $mg = P$, whence

$$m = \frac{P}{g} \ . \tag{1.3}$$

Thus, the mass of a body is equal to its weight divided by the acceleration of gravity g.

The Third Law (the Law of Action and Reaction) establishes the character of mechanical interaction between material bodies. For two particles it states: *two particles exert on each other forces equal in magnitude and acting in opposite directions along the straight line connecting the two particles.*

It should be noted that the forces of interaction between free particles (or bodies) do not form a balanced system, as they act on different objects.

The third law of dynamics, which establishes the character of interaction of material particles, plays an important part in the dynamics of systems.

1.3. The Problems of Dynamics for a Free and a Constrained Particle. The problems of dynamics for a free particle are: 1) knowing the equation of motion of a particle, to determine the force acting on it (the first problem of dynamics), 2) knowing the forces acting on a particle, to determine its equation of motion (the second, or principal, problem of dynamics).

Both problems are solved with the help of Eq. (1.1) or (1.2), which express the fundamental law of dynamics, since they give the relation between acceleration, i.e., the quantity characterizing the motion of a particle, and the forces acting on it.

In engineering it is often necessary to investigate constrained motions of a particle, i.e., cases when constraints attached to a particle compel it to move along a given fixed surface or curve.

In such cases we shall use, as in statics, the axiom of constraints, which states that any constrained particle can be treated as a free body detached from its constraints provided the latter is represented by their reactions $\vec{N}$. Then the fundamental law of dynamics for the constrained motion of a particle takes the form

$$m\vec{w} = \sum \vec{F}^a{}_k + \vec{N} \, , \tag{1.4}$$

where $\vec{F}^a{}_k$ denotes the applied forces acting on the particle.

For constrained motion, the first problem of dynamics will usually be: to determine the reactions of the constraints acting on a particle if the motion and applied forces are known. The second (principal) problem of dynamics for such motion will pose two questions: knowing the applied forces, to determine: a) the equation of motion of the particle and b) the reaction of its constraints.

2. DIFFERENTIAL EQUATIONS OF MOTION FOR A PARTICLE AND THEIR INTEGRATION

2.1. **Rectilinear Motion of a Particle**. We know from kinematics that in rectilinear motion the velocity and acceleration of a particle are continuously directed along the same straight line. As the direction of acceleration is coincident with the direction of force, it follows that a free particle will move in a straight line whenever the force acting on it is of constant direction and the velocity at the initial moment is either zero or is collinear with the force.

Consider a particle moving rectilinearly under the action of an applied force $\vec{R} = \Sigma\vec{F}_k$. The position of the particle on its path is specified by its coordinate x (Fig. 1). In this case the principal problem of dynamics is, knowing $\vec{R}$, to find the equation of motion of the particle $x=f(t)$. Eq. (1.2) gives the relation between x and $\vec{R}$. Projecting both sides of the equation on axis Ox, we obtain $mw_x=R_x\equiv\Sigma F_{kx}$ or as, $w_x = \frac{d^2x}{dt^2}$,

$$m\frac{d^2x}{dt^2} = \sum F_{kx} . \tag{2.1}$$

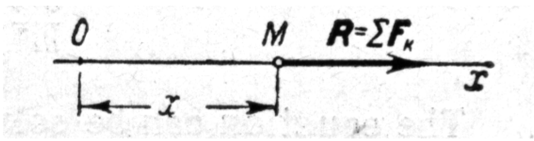


Fig.1.

Eq. (2.1) is called the differential equation of rectilinear motion of a particle. It is often more convenient to replace Eq. (2.1) with two differential equations containing first derivatives:

$$m\frac{dv_x}{dt} = \sum F_{kx} \tag{2.2}$$

$$\frac{dx}{dt} = v_x. \tag{2.2'}$$

Whenever the solution of a problem requires that the velocity be found as a function of the coordinate x instead of time t (or when the forces themselves depend on x), Eq. (2.2') is converted to the variables x. As $\frac{dv_x}{dt} = \frac{dv_x}{dx} \times \frac{dx}{dt} = \frac{dv_x}{dx} v_x$ Eq. (2.2) takes the form

$$mv_x\frac{dv_x}{dx} = \sum F_{kx}. \tag{2.3}$$

The principal problem of dynamics is, essentially, to develop the equation of motion $x=f(t)$ for a particle from the above equations, the forces being known. For this it is necessary to integrate the corresponding differential equation. In order to make clearer the nature of the mathematical problem, it should be recalled that the forces in the right side of Eq. (2.1) can depend on time *t*, on the position of the particle *x*, or on the

velocity $v_x = \frac{dx}{dt}$. Consequently, in the general case Eq. (2.1) is, mathematically, a differential equation of the second order in the form

$$\frac{d^2x}{dt^2} = \Phi(t, x, \frac{dx}{dt}). \qquad (2.4)$$

The equation can be solved for every specific problem after determining the form of its right-hand member, which depends on the applied forces. When Eq. (2.4) is integrated for a given problem, the general solution will include two constants of integration C_1 and C_2 and the general form of the solution will be

$$x=f(t, C_1, C_2). \qquad (2.5)$$

To solve a concrete problem, it is necessary to determine the values of the constants C_1 and C_2. For this we introduce the so-called initial conditions.

Investigation of any motion begins from some specified instant called the initial time $t=0$, usually the moment when the motion under the action of the given forces starts. The position occupied by a particle at the initial time is called its initial displacement, and its velocity at that time is its initial velocity (a particle can have an initial velocity either because at time $t=0$ it was moving under no force or because up to time $t=0$ it was subjected to the action of some other forces). To solve the principal problem of dynamics we must know, besides the applied forces, the initial conditions, i.e., the position and velocity of the particle at the initial time.

In the case of rectilinear motion, the initial conditions are specified in the form

$$at\ t=0,\ x=x_0,\ v_x = v_0. \qquad (2.6)$$

From the initial conditions we can determine the meaning of the constants C_1 and C_2, and develop finally the equation of motion for the particle in the form

$$x = f(t, x_0, v_0,). \qquad (2.7)$$

The following simple example will explain the above. Let there be acting on a particle a force *Q* of constant magnitude and direction. Then Eq. (2.2) acquires the form

$$m\frac{dv_x}{dt} = Q_x.$$

As $Q_x = const.$, multiplying both members of the equation by *dt* and integrating, we obtain

$$v_x = \frac{Q_x}{m}t + C_1. \qquad (2.8)$$

Substituting the value of v_x into Eq. (2.2'), we have

$$\frac{dx}{dt} = \frac{Q_x}{m}t + C_1.$$

Multiplying through by *dt* and integrating once again, we obtain

$$x = \frac{1}{2}\frac{Q_x}{m}t^2 + C_1 t + C_2. \qquad (2.9)$$

This is the general solution of Eq. (2.4) for the specific problem in the form given by Eq. (2.5).

Now let us determine the integration constants C_1 and C_2, assuming for the specific problem the initial conditions given by (2.6). Solutions (2.8) and (2.9) must satisfy any moment of time, including $t=0$. Therefore, substituting zero for t in Eqs. (2.8) and (2.9), we should obtain v_0 and x_0, instead of v_x and x, i.e., we should have

$$v_0 = C_1, \quad x_0 = C_2.$$

These equations give the values of the constants C_1 and C_2, which satisfy the initial conditions of a given problem. Substituting these values into Eq. (2.9), we obtain finally the relevant equation of motion in the form expressed by Eq. (2.7):

$$x = x_0 + v_0 t + \frac{1}{2}\frac{Q_x}{m}t^2. \qquad (2.10)$$

We see from Eq. (2.10) that a particle subjected to a constant force performs uniformly variable motion. This could have been foreseen; for, if $Q=$ *const.*, $w=$ *const.*, too. An example of this type of motion is the motion of a particle under the force of gravity, in which case in Eq. (2.10) $\frac{Q_x}{m} = g$ and axis Ox is directed vertically down.

2.2. Curvilinear Motion of a Particle. Consider a free particle moving under the action of forces $\vec{F}_1$, $\vec{F}_2$,... $\vec{F}_n$. Let us draw a fixed set of axes *Oxyz* (Fig. 2). Projecting both members of the equation $m\vec{w} = \sum \vec{F}_k$ on these axes, and taking into account that $w_x = \frac{d^2x}{dt^2}$ we obtain the differential equations of curvilinear motion of a body in terms of the projections on rectangular cartesian axes:

$$m\frac{d^2x}{dt^2} = \sum F_{kx}, \quad m\frac{d^2y}{dt^2} = \sum F_{ky}, \quad m\frac{d^2z}{dt^2} = \sum F_{kz}. \qquad (2.11)$$

As the forces acting on the particle may depend on time, the displacement or the velocity of the particle, then by analogy with Eq. (2.4), the right-hand members of Eq. (2.11) may contain the time *t*, the coordinates *x, y, z* of the particle, and the projections of its velocity $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Furthermore, the right side of each equation may include all these variables.

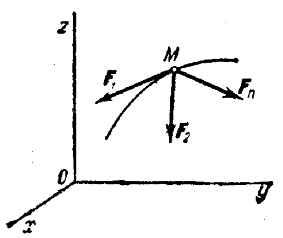


Fig.2.

Eq. (2.11) can be used to solve both the first and the second (the principal) problems of dynamics. To solve the principal problem of dynamics we must know, besides the acting forces, the initial conditions, i.e., the position and velocity of the particle at the initial time. The initial conditions for a set of coordinate axes $Oxyz$ are specified in the form: *at t=0,*

$$x = x_0, \quad y = y_0, \quad z = z_0$$

$$\upsilon_x = \upsilon_{x0}, \upsilon_y = \upsilon_{y0}, \upsilon_z = \upsilon_{z0} \qquad (2.12)$$

Knowing the acting forces, by integrating Eq. (2.11) we find the coordinates *x, y, z* of the moving particle as functions of time *t,* i.e., the equation of motion for the particle. The solutions will contain six constants of integration C_1, C_2, ... C_6, the values of which must be found from the initial condition (2.12). An example of integrating Eqs. (2.11) is given in §.2.3.

2.3. Motion of a Particle Thrown at an Angle to the Horizon in a Uniform Gravitational Field. Let us investigate the motion of a projectile thrown with an initial velocity $\vec{v}_0$ at an angle α to the horizon, considering it as a material particle of mass *m,* neglecting the resistance of the atmosphere, assuming that the horizontal range is small as compared with the radius of the earth and considering the gravitational field to be uniform *(P= const.).*

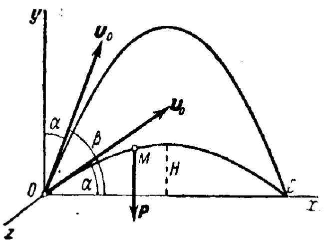


Fig.3.

Place the origin of the coordinate axes *0* at the initial position of the particle, direct the y-axis vertically up, the *x* axis in the plane through *Oy* and vector $\vec{v}_0$, and the *z*-axis perpendicular to the first two (Fig. 3). The angle between vector $\vec{v}_0$ and the *x*-axis will be α.

Draw now moving particle *M* anywhere on its path. Acting on the particle is only the force of gravity $\vec{P}$, the projections of which on the coordinate axes are $P_x=0$, $P_y=-P=-mg$, $P_z=0$.

Substituting these values into Eq. (2.11) and noting that $\frac{d^2x}{dt^2} = \frac{dv_x}{dt}$, etc., after eliminating *m* we obtain:

$$\frac{d\upsilon_x}{dt} = 0, \quad \frac{d\upsilon_y}{dt} = 0, \quad \frac{d\upsilon_z}{dt} = 0.$$

Multiplying these equations by *dt* and integrating, we find $\upsilon_x=C_1$, $\upsilon_y=-gt+C_2$, $\upsilon_z=C_3$.

The initial conditions of our problem have the form:

$$at\ t=0, \qquad x=0, \qquad y=0, \qquad z=0;$$

$$\upsilon_x = \upsilon_0 \cos\alpha, \quad \upsilon_y = \upsilon_0 \sin\alpha, \quad \upsilon_z = 0.$$

Satisfying the initial conditions, we have

$$C_1=\upsilon_0 \cos\alpha, \ C_2 = \upsilon_0 \sin\alpha, \ C_3 = 0.$$

Substituting these values of C_1, C_2 and C_3, in the solutions above and replacing υ_x, υ_y, υ_z by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, we arrive at the equations

$$\frac{dx}{dt} = \upsilon_0 \cos\alpha, \quad \frac{dy}{dt} = \upsilon_0 \sin\alpha - gt, \quad \frac{dz}{dt} = 0.$$

Integrating, we obtain $x=\upsilon_0 t\cos\alpha + C_4,\ y=\upsilon_0 t\sin\alpha - \frac{gt^2}{2} + C_5,\ z=C_6.$

Substituting the initial conditions, we have $C_1=C_2=C_3=0$. And finally we obtain the equations of motion of particle M in the form

$$x=\upsilon_0 t\cos\alpha,\ \ y=\upsilon_0 t\sin\alpha - \frac{gt^2}{2} C_5,\ \ z=0. \tag{2.13}$$

From the last equation it follows that the motion takes place in the plane O_{xy}.

Knowing the equations of motion of a particle it is possible to determine all the characteristics of the given motion by the methods of kinematics.

1. Path. Eliminating the time t between the first two of Eqs. (2.13), we obtain the equation of the path of the particle:

$$y=x\tan\alpha - \frac{gx^2}{2\upsilon_0^2\cos^2\alpha}\ . \tag{2.14}$$

This is an equation of a parabola the axis of which is parallel to the y- axis. Thus, a heavy particle thrown at an angle to the horizon in vacuum follows a parabolic path.

2. Horizontal Range. The horizontal range is the distance $OC=X$ along the x-axis. Assuming in Eq. (2.14) $y=0$, we obtain the points of intersection of the path with the x- axis. From the equation

$$x\left(\tan\alpha - \frac{gx}{2\upsilon_0^2\cos^2\alpha}\right)=0$$

we obtain

$$x_1=0,\quad x_2=\frac{2\upsilon_0^2\cos^2\alpha\cdot\tan\alpha}{g}.$$

The first solution gives point 0, the second point C. Consequently $X=x_2$ and finally

$$X=\frac{\upsilon_0^2}{g}\sin 2\alpha. \tag{2.15}$$

From Eq. (2.15) we see that the horizontal range X is the same for angle β, where $2\beta = 180° - 2\alpha$, i.e., if $\beta=90°-\alpha$. Consequently, a particle thrown with a given initial velocity υ_0 can reach the same point C by two paths: flat (low) $(\alpha<45°)$ or curved (high) $(\beta=90°-\alpha>45°)$. With a given initial velocity υ_0 the maximum horizontal range in vacuum is obtained when $\sin 2\alpha=1$, i.e., when angle $\alpha=45°$.

3. Height of path. If in Eq. (2.14) we assume $x=\frac{1}{2}X=\frac{\upsilon_0^2}{g}\sin\alpha\cos\alpha$, we obtain the height H of the path:

$$H=\frac{\upsilon_0^2}{g}\sin^2\alpha. \tag{2.16}$$

4. Time of flight. It follows from Eq. (2.13) that the total time of flight is defined by the equation $X=\upsilon_0 T\cos\alpha$. Substituting the expression for X, we obtain

$$T = \frac{2v_0^2}{g}\sin\alpha. \qquad (2.17)$$

At the maximum range angle $\alpha^*=45°$, all the quantities become respectively

$$X^* = \frac{v_0^2}{g}, \ T^* = \frac{v_0^2}{g}\sqrt{2}, \quad H^* = \frac{v_0^2}{4g} = \frac{1}{4}X^*.$$

3.VIBRATION OF A PARTICLE

3.1. Free Harmonic Motion. The study of vibrations is essential for a number of physical and engineering fields. Although the vibrations studied in such different fields as mechanics, radio engineering, and acoustics are of different physical nature, the fundamental laws hold good for all of them. The study of mechanical vibrations is therefore of importance not only because they are frequently encountered in engineering but also because the results obtained in investigating mechanical vibrations can be used in studying and understanding vibration phenomena in other fields.

We shall start with examining free harmonic motion of a particle. Consider a particle *M* (Fig. 4) moving rectilinearly under the action of a *restoring force* $\vec{F}$ directed towards a fixed centre *0* and proportional to the distance from that centre. The projection of $\vec{F}$ on the axis Ox is

$$F_x = -cx. \qquad (3.1)$$

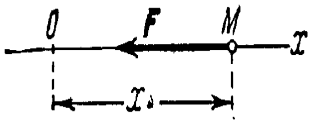


Fig.4.

We see that the force $\vec{F}$ tends to return the particle to its position of equilibrium *0,* where $\vec{F}=0$, which is why it is called a "restoring" force. Let us derive the equation of motion of the particle *M.* Writing the differential equation of motion (2.1), we obtain

$$m\frac{d^2x}{dt^2} = -cx.$$

Dividing both sides of the equation by *m* and introducing notation

$$\frac{c}{m} = k^2, \qquad (3.2)$$

we reduce the equation to the form

$$\frac{d^2x}{dt^2} + k^2x = 0. \qquad (3.3)$$

Eq. (3.3) is the *differential equation of free harmonic motion.* Referring to the theory of differential equations, as the roots of a characteristic equation of the type of Eq. (3.3) are imaginary, its general solution will be

(3.4)

$$x = C_1 \sin kt + C_2 \cos kt$$

where C_1 and C_2 are constants of integration.

If we replace C_1 and C_2 by constants a and α, such that $C_1 = a\cos\alpha$ and $C_2 = a\sin\alpha$, we obtain

$$x = a(\sin kt \times \cos\alpha + \cos kt \times \sin\alpha), \text{ or}$$

$$x = a\sin(kt + \alpha). \tag{3.5}$$

This is another form of the solution of Eq. (3.3) in which the constants of integration appear as a and α, and which is more convenient for general analyses.

The velocity of a particle in this type of motion is

$$\upsilon_x = \frac{dx}{dt} = ak\cos(kt+\alpha). \tag{3.6}$$

The vibration of a particle described by Eq. (3.5) is called *simple harmonic motion.*

The quantity a, which is the maximum distance of M from the centre of vibration, is called the *amplitude of vibration.* The quantity $\varphi = kt + \alpha$ is called the *phase of vibration.* Unlike the coordinate x, the phase φ defines both the position of the particle at any given time and the direction of its subsequent motion.

The quantity k is called the *angular, or circular, frequency of vibration.*

The time T in which the moving particle makes one complete oscillation is called the *period of vibration.* In one period the phase changes by 2π. Consequently, we must have $kT = 2\pi$, whence the period

$$T = \frac{2\pi}{k}. \tag{3.7}$$

The quantity ν, which is the inverse of the period and specifies the number of oscillations per second, is called the *frequency of vibration:*

$$\nu = \frac{1}{T} = \frac{k}{2\pi}.$$

It can be seen from this that the quantity k differs from ν only by a constant multiplier 2π. Usually we shall speak of the quantity k as of frequency.

The values of a and $\boldsymbol{\alpha}$ are determined from the initial conditions. Assuming that, at $t = 0$, $x = x_0$ and $\upsilon_x = \upsilon_0$ we obtain from Eqs. (3.5) and (3.6) $x_0 = a\sin\alpha$ and $\frac{\upsilon_0}{k} = a\cos\alpha$. By first squaring and adding these equations and then dividing them, we obtain

$$a = \sqrt{x_0^2 + \frac{\upsilon_0^2}{k}}, \quad \tan\alpha = \frac{kx_0}{\upsilon_0}.$$

Note the following properties of free harmonic motion:

1) The amplitude and initial phase *depend* on the initial conditions;

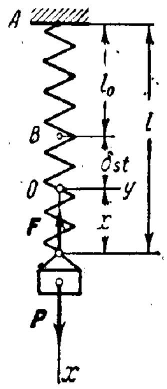
2) The frequency k, and consequently the period T, *do not depend* on the initial conditions and are invariable characteristics for a given vibrating system.

It follows, in particular, that if a problem requires that only the period (or frequency) of vibration be determined, it is necessary to write a differential equation of motion in the form (3.3). Then T is found immediately from Eq. (3.7) without integrating.

Consider the next problem: A weight is attached to end B of a vertical spring AB and released from rest. Determine the law of motion of the weight if the elongation of the spring in the equilibrium condition is δ_{st} (the static elongation of the spring).

Solution. Place the origin 0 of the coordinate axis in the position of static equilibrium of the system and direct the axis Ox vertically down. The elastic force $F = c|\Delta l|$. In our case $\Delta l = \delta_{st} + x$, hence

$$F_x = -c(\delta_{st} + x).$$



Writing the differential equation of motion, we obtain

$$m\frac{d^2x}{dt^2} = -c(\delta_{st} + x) + P.$$

But from the conditions of the problem the gravitational force $P = mg = c\delta_{st}$ (in the position of equilibrium force P is balanced by the elastic force $c\delta_{st}$). Introducing the notation $\frac{c}{m} = \frac{g}{\delta_{st}} = k^2$, we reduce the equation to form

$$\frac{d^2x}{dt^2} + k^2x = 0,$$

whence immediately we find the period of vibration:

$$T = \frac{2\pi}{k} = 2\pi\sqrt{\frac{\delta_{st}}{g}}.$$

Thus, the *period of vibration is proportional to the square root of the static elongation of the spring* (this holds good also for a load vibrating on an elastic beam, where δ_{st} *is* the static deflection of the beam).

The solution of the obtained differential equation is

$$x = C_1 \sin kt + C_2 \cos kt.$$

From the initial conditions, *at* $t = 0$, $x = -\delta_{st}$, *and* $\upsilon_x = 0$. As

$$\upsilon_x = \frac{dx}{dt} = kC_1 \cos kt - kC_2 \sin kt,$$

substituting the initial conditions, we obtain $C_2 = -\delta_{st}$, $C_1 = 0$. Hence, the amplitude of vibration is δ_{st} and the motion is according to the law

$$x = -\delta_{st} \cos kt.$$

We see that the maximum elongation of the spring in this motion is $2\delta_{st}$.

This solution shows that *a constant force P does not change the type of motion under the action of an elastic force F but only shifts the center of the vibrations in the direction of the action of the force by the quantity* δ_{st} (without the force P the vibration would, evidently, be about B).

3.2. Damped Vibration. Let us see how the resistance of a surrounding medium affects vibrations, assuming the resisting force proportional to the first power of the velocity: $\vec{R} = -\mu\vec{v}$ (the minus indicates that force $\vec{R}$ is opposite to $\vec{v}$). Let a moving particle be acted upon by a restoring force $\vec{F}$ and a resisting force $\vec{R}$ (Fig.5).

Fig.5.

Then $F_x = -cx$, $R_x = -\mu v_x = -\mu\dfrac{dx}{dt}$ and the differential equation of motion is

$$m\frac{d^2x}{dt^2} = -cx - \mu\frac{dx}{dt}.$$

Dividing both sides by m, we obtain

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0 \qquad \textbf{(3.8)}$$

where

$$\frac{c}{m} = k^2, \quad \frac{\mu}{m} = 2b. \qquad \textbf{(3.9)}$$

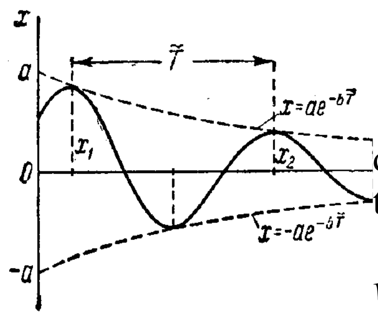


Fig.6.

It is easy to verify that k and b have the same dimension (sec^{-1}), which makes it possible to compare them.

Eq. (3.8) is called the *differential equation of damped vibration.* The solution of Eq (3.8) can be found by passing to a new variable z through the equality $x = ze^{-bt}$. Then

$$\frac{dx}{dt} = e^{-bt}\left(\frac{dz}{dt} - bz\right); \frac{d^2x}{dt^2} = e^{-bt}\left(\frac{d^2z}{dt^2} - 2b\frac{dz}{dt} + b^2z\right).$$

Substituting these expressions and the expression of x into Eq. (3.8), and after the necessary computation, we obtain

$$\frac{d^2z}{dt^2} + (k^2 - b^2)z = 0. \qquad \textbf{(3.10)}$$

Let us consider the case when $k>b$, i.e., when the resistance is small as compared with the restoring force. Introducing the notation

$$\tilde{k} = \sqrt{k^2 - b^2} \quad \textbf{(3.11)}$$

we see that Eq. (3.10) coincides with Eq. (3.3).Consequently, $z = a\ sin(\tilde{k}t+\alpha)$ or, passing to x,

$$x = ae^{-bt} sin\ (\tilde{k}t+\alpha). \quad \textbf{(3.12)}$$

The expression (3.12) gives the solution of differential equation (3.8). The quantities a and α are constants of integration and are determined by the initial conditions.

Vibrations according to the law (3.12) are called *damped* because, due to the multiplier e^{-bt}, the value of x decreases with time and tends to zero. A graph of such vibrations is given in Fig. 6. The graph shows that the vibrations are not periodic, though they do show a certain repetition. For example, a particle oscillating about a centre *0* returns to that centre at certain intervals $\tilde{T}$ equal to the period $sin(\tilde{k}t+\alpha)$.

Therefore the quantity

$$\tilde{T} = \frac{2\pi}{\tilde{k}} = \frac{2\pi}{\sqrt{\tilde{k}^2 - b^2}} \quad \textbf{(3.13)}$$

is conventionally called the *period of damped vibration.* Comparing, Eqs. (3.13) and (3.7), we see that $\tilde{T} > T$ i.e., that resistance to vibration tends to increase the period of the vibration. When however, the resistance is small $(b\langle\langle k)$ the quantity b^2 can be neglected in comparison with $\tilde{k}^2$ and we can assume $\tilde{T} \approx T$. Thus a small resistance has no practical effect on the period of vibration.

The time interval between two successive displacements of an oscillating particle to the right or to the left is also equal to $\tilde{T}$. Hence, if the maximum displacement x, to the right takes place at time t_1 the second displacement x_2 will be at time $t_2 = t_1+\tilde{T}$, etc. Then, by Eq. (3.12) and taking into account that $\tilde{k}\tilde{T} = 2\pi$, we have

$$x_1 = ae^{-bt_1} \sin\ (\tilde{k}t_1 + \alpha),$$

$$x_2 = ae^{-b(t_1+\tilde{T})} \sin\ (\tilde{k}t_1 + \tilde{k}\tilde{T} + \alpha) = x_1 e^{-b\tilde{T}}.$$

Similarly, for any displacement x_{n+1} we will have $x_{n+1} = x_1 e^{-b\tilde{T}}$. Thus we find that the amplitude of vibration decreases in geometric progression. The denominator of this progression $e^{-b\tilde{T}}$ is called the *damping decrement,* and the modulus of its logarithm, i.e., the quantity $b\tilde{T}$, the *logarithmic decrement.*

It follows from these results that a small resistance has practically no effect on the period of vibration, but gradually damps it by virtue of the amplitude of vibration decreasing according to a law of geometric progression.

When the resistance is large and $b > k$, the solution of Eq. (3.10) contains no trigonometric functions. The particle no longer oscillates but instead, under the influence of the restoring force, gradually approaches the position of equilibrium.

3.3. Damped Forced Vibrations. Resonance. Consider the motion of a particle on which are acting a restoring force $\vec{F}$, a damping force $\vec{R}$ proportional to the velocity (see § 3.2), and a disturbing force $\vec{Q}$**,** whose projection on the axis Ox is $Q_x = Q_0 \sin pt$. The differential equation of this motion has the form

$$m\frac{d^2x}{dt^2} = -cx - \mu\frac{dx}{dt} + Q_0 \sin pt.$$

Dividing both sides of the equation by m, assuming $\frac{Q_0}{m} = P_0$ and taking into account the expression (3.9), we obtain

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = P_0 \sin pt \quad . \qquad \textbf{(3.14)}$$

Eq. (3.14) is the *differential equation of damped forced vibration of a particle.* Its general solution, as is known, has the form $x = x_1 + x_2$, where x_1 is the general solution of the equation without the right side, i.e., of Eq. (3.8) [at $k>b$ this solution is given by Eq. (3.12)], and x_2, is a particular solution of the complete equation (3.14). Let us find the solution x_2 in the form

$$x_2 = A \sin(pt - \beta),$$

where A and β are constants so chosen that Eq. (3.14) should become an identity. Differentiating, we obtain

$$\frac{dx_2}{dt} = Ap\cos(pt - \beta), \quad \frac{d^2x_2}{dt^2} = -Ap^2 \sin(pt - \beta).$$

Substituting these expressions of the derivatives and x_2 into the left side of Eq. (3.14) and introducing for the sake of brevity the notation $pt - \beta = \psi$ (or $pt = \psi + \beta$), we obtain

$$A(-p^2+k^2) \sin \psi + 2bpA \cos \psi = P_0 (\cos \beta \sin \psi + \sin \beta \cos \psi).$$

For this equation to be satisfied at any value of ψ, i.e., at any instant of time, the factors of $\sin \psi$ and $\cos \psi$ in the left and right sides should be separately equal. Hence,

$$A\,(k^2 - p^2) = P_0 \cos\beta,\ 2bpA = P_0 \sin\beta.$$

First squaring and adding these equations, and then dividing one by the other, we obtain:

$$A = \frac{P_0}{\sqrt{(k^2 - p^2)^2 + 4b^2p^2}},\ \tan\beta = \frac{2bp}{k^2 - p^2}. \qquad \textbf{(3.15)}$$

As $x = x_1 + x_2$ and the expression x_1 is given by Eq. (3.12) we have the final solution of Eq. (3.14) in the form

$$x = ae^{-bt}\sin(\tilde{k}t + \alpha) + A\sin(pt - \beta). \qquad \textbf{(3.16)}$$

Here a and α are constants of integration determined from the initial conditions, and the expressions for A and β are given by Eqs. (3.15) and do not depend on the initial conditions. These vibrations are compounded of *natural* vibration [the first term in Eq. (3.16); Fig. 7 *a*] and *forced* vibration [the second term in Eq. (3.16); Fig. 7 *b*]. The natural vibration of the particle in such a case was discussed in § 3.2. It was established that it is transient and is damped fairly quickly, and after a certain interval of time t_t called the *transient period,* can be neglected. A curve showing the transient vibration is given in Fig. 7 *c*. For practical purposes it can thus be assumed that after a certain transient period a particle will vibrate according to the law

$$x = A\sin(pt - \beta). \qquad \textbf{(3.17)}$$

This is steady-state *forced vibration,* a sustained periodic motion with amplitude A defined by Eq. (3.15) and a frequency p equal to the impressed frequency. The quantity β characterizes the *phase shift* of forced vibration with respect to the disturbing force. Let us investigate the results obtained. First let us introduce the notation

$$\frac{p}{k} = \lambda,\ \frac{b}{k} = h,\ \frac{P_0}{k^2} = \frac{Q_0}{c} = \delta_0, \qquad \textbf{(3.18)}$$

where λ is the frequency ratio, h a quantity characterizing the damping effect, δ_0 the magnitude of the static deflection of a particle under the action of force Q_0.

Then, dividing the numerator and denominator of Eq. (3.15) by k^2, we obtain

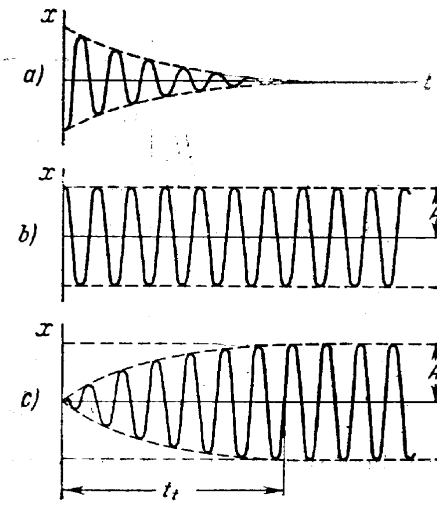


Fig.7.

$$A = \frac{\delta_0}{\sqrt{(1-\lambda^2)^2 + 4h^2\lambda^2}},\ \tan\beta = \frac{2h\lambda}{1-\lambda^2}. \qquad \textbf{(3.19)}$$

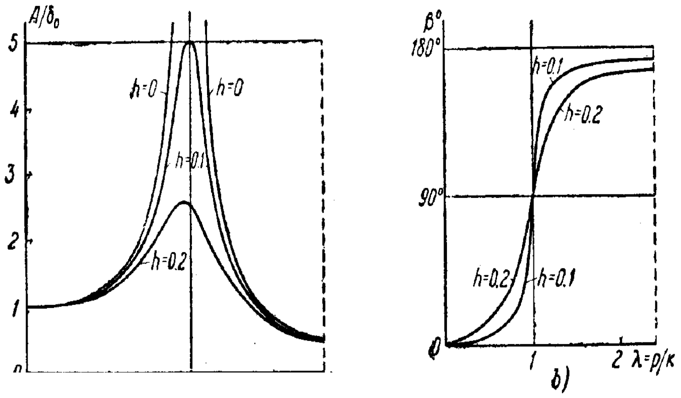


Fig.8.

It can be seen from Eq. (3.19) that A and β depend on two dimensionless parameters λ and h. Graphs of this relation for certain values of h are given in Fig. 8. The values of δ_0, λ and h can be computed for each specific problem from its conditions, and the values of A and p determined from the respective graphs or Eqs. (3.19).

These graphs (and equations) also show that by altering the frequency ratio λ we can induce forced vibrations of different amplitude.

When the resistance is very small (as ordinarily in the atmosphere) and λ is not close to unity, it is possible in Eqs. (3.19) to assume approximately $h \approx 0$. In this case we obtain

$$A \approx \frac{\delta_0}{\left|1-\lambda^2\right|}\,;\ \beta \approx 0 (at \lambda \langle 1),\ \beta \approx 180\ (at\, \lambda \rangle 1)\,.$$

Let us consider also the following special cases. 1) If the frequency ratio λ is very small$(p \langle\langle\, k)$, then, assuming as an approximation $\lambda \approx 0$, we obtain from Eq. (3.19)

$A \approx \delta_0$.

The vibration in this case has an amplitude equal to the static deflection δ_0 and the phase shift is $\beta = 0$.

2) If the frequency ratio λ is very large $(p \rangle\rangle\, k)$, A becomes very small. This case is of special interest for the absorption of vibrations in structures, instruments, etc. Assuming the resistance to be small and neglecting $2h\lambda$ and 1 as compared with λ^2 in Eq. (3.19), we obtain for computing A an approximate formula:

$$A = \frac{\delta}{\lambda^2} = \frac{P_0}{p^2}.$$

3) In all cases of practical interest h is very small. Then, from Eq. (3.19), if λ is almost unity the amplitude of forced vibrations becomes very large. This phenomenon is called *resonance.*

At resonance we can assume $\lambda = 1$ in Eq. (3.19), and then

$$A_r = \frac{\delta_0}{2h},\ \beta_r = \frac{\pi}{2}. \qquad \textbf{(3.20)}$$

We see that when h is small A_r can become very large.

When the damping force, and with it h, tends to zero, the limiting value of the amplitude A_r as Eq. (3.20) shows, tends to infinity. Thus, with no damping force the vibration amplification process in resonance conditions is unlimited and the amplitude increases indefinitely. A graph of resonance vibration is given in Fig. 9. When the damping forces are very small the picture is similar.

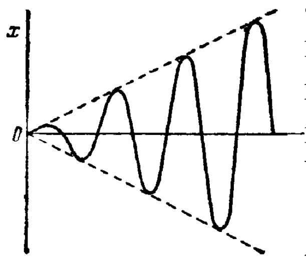


Fig.9.

General Properties of Forced Vibration. It follows from the results obtained above that forced vibration has the following important properties, which distinguish it from the natural vibration of a particle:

1) The amplitude of forced vibration does not depend on the initial conditions. 2) Forced vibration does not die out in the presence of resistance. 3) The frequency of forced vibration is equal to the frequency of the disturbing force and does not depend on the characteristics of the vibrating system (the disturbing force "impresses" its own vibration frequency on the system). 4) Even when the disturbing force Q_o is small, large forced vibration can be induced if the resistance is small and the frequency p is almost equal to k (resonance). 5) Even if the disturbing force is large, forced vibration can be damped if the frequency p is much larger than k.

Forced vibration, and resonance in particular, plays an important part in many branches of physics and engineering. Lack of balance in working machines and motors, for example, usually causes forced vibration to appear in the machine or its foundation.

In radio engineering the reverse is true. Resonance is extremely useful and is used to separate the signals of one radio station from those of all others (tuning).

4. INTRODUCTION TO THE DYNAMICS OF A SYSTEM

4.1. Mechanical Systems. External and Internal Forces. A mechanical system is defined as such a collection of material points (particles) or bodies in which the position or motion of each particle or body of the system depends on the position and motion of all the other particles or bodies. We shall thus regard a material body as a system of its particles.

A classical example of a mechanical system is the solar system, all the component bodies of which are connected by the forces of their mutual attraction.

A collection of bodies not connected by interacting forces does not comprise a mechanical system (e.g., a group of flying aircraft). In this summary we shall consider only mechanical systems, calling them just "systems" for short.

The forces acting on the particles or bodies of a system can be subdivided into external and internal forces.

External forces are defined as the forces exerted on the members of a system by particles or bodies not belonging to the given system. Internal forces are defined as the forces of interaction between the members of the same system. We shall denote external forces by the symbol $\vec{F}^e$, and internal forces by the symbol $\vec{F}^i$. Both external and internal forces can be either active forces or the reactions of constraints. The division of forces into external and internal is purely relative, and it depends on the extent of the system whose motion is being investigated. In considering the motion of the solar system as a whole, for example, the gravitational attraction of the sun acting on the earth is an internal force; in investigating the earth's motion about the sun, the same force is external.

Internal forces possess the following properties:

1. The geometrical sum (the principal vector) of all the internal forces of a system is zero. This follows from the third law of dynamics, which states that any two particles of a system (Fig.10) act on each other with equal and oppositely directed forces $\vec{F}^i{}_{12}$ and $\vec{F}^i{}_{21}$, the sum of which is zero. Since the same is true for any pair of particles of a system, then

$$\sum \vec{F}^i{}_k = 0 \,. \qquad \textbf{(4.1)}$$

2. The sum of the moments (the principal moment) of all the internal forces of a system with respect to any centre or axis is zero. For if we take an arbitrary centre 0, it is apparent from Fig.10 that $\vec{m}_0(\vec{F}^i{}_{12}) + \vec{m}_0(\vec{F}^i{}_{21}) = 0$. The same result holds good for the moments about any axis. Hence, for the system as a whole we have

$$\sum \vec{m}_0(\vec{F}^i{}_k) = 0 \text{ or} \sum m_x(\vec{F}^i{}_k) = 0. \qquad (4.2)$$

It does not follow from the above, however, that the internal

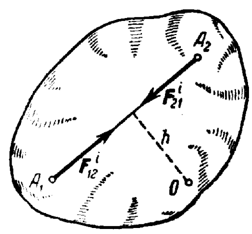


Fig.10.

forces are mutually balanced and do not affect the motion of the system, for they are applied to different particles or bodies and may cause their mutual displacement. The internal forces will be balanced only when a given system is a rigid body.

4.2. Mass of a System. Centre of Mass. The motion of a system depends, besides the acting forces, on its total mass and the distribution of this mass. The *mass of a system* is equal to the arithmetical sum of the masses of all the particles or bodies comprising it:

$$M = \sum m_k . \qquad \textbf{(4.3)}$$

The distribution of mass is characterized primarily by the location of a point called the centre of mass. *The centre of mass or centre of inertia, of a system is defined as a geometrical point C whose coordinates are given by the equations:*

$$x_C = \frac{\sum m_k x_k}{M},\ y_C = \frac{\sum m_k y_k}{M},\ z_C = \frac{\sum m_k z_k}{M}. \qquad \textbf{(4.4)}$$

where m_k is the mass of a particle of the system, and x_k, y_k,z_k are its coordinates.
If the position of a centre of mass is defined by its radius vector $\vec{r}_c$, we can obtain from Eqs. (4.4) the following expression

$$\vec{r}_c = \frac{\sum m_k \vec{r}_k}{M}, \qquad \textbf{(4.5)}$$

where $\vec{r}_k$ is the radius vector of a particle of the system.

For a body in a uniform gravitational field, the centre of mass coincides with the centre of gravity. The concepts of centre of gravity and centre of mass, however, are not identical. The concept of centre of gravity, as the point through which the resultant of the forces of gravity passes, has meaning only for a rigid body in a uniform field of gravity. The concept of centre of mass, as a characteristic of the distribution of mass in a system, on the other hand, has meaning for any system of particles or bodies, regardless of whether a given system is subjected to the action of forces or not.

4.3. Moment of Inertia of a Body About an Axis. Radius of Gyration. The position of centre of mass does not characterise completely the distribution of mass in a system. For if in the system in Fig.11 the distance h of each of two identical spheres A and B from the axis Oz is increased by the same quantity, the location of the centre of mass will not change, though the distribution of mass will change and influence the motion of the system (all other conditions remaining the same, the rotation about axis Oz will be slower).

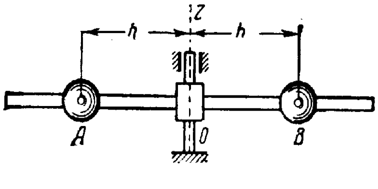


Fig.11.

Accordingly, another characteristic of the distribution of mass, called the moment of inertia, is introduced in mechanics. *The moment of inertia of a body with respect to a given axis Oz is defined as a scalar quantity equal to the sum of the masses of the particles of the body, each multiplied by the square of its perpendicular distance from the axis:*

$$J_z = \sum m_k h_k^2. \quad \textbf{(4.6)}$$

It will be shown further on that moment of inertia plays the same part in the rotational motion of a body as mass does in translatory motion, i.e., *moment of inertia is a measure of a body's inertia in rotational motion.*

By Eq. (4.6), the moment of inertia of a body is equal to the sum of the moments of inertia of all its parts with respect to the same axis. For a material point located at a distance h from an axis, $J_z = mh^2$. The dimension of moment of inertia in the technical system of units is [J] = kgm-sec^2.

The concept of *radius of gyration* is often employed in calculations. The radius of gyration of a body with respect to an axis Oz is a linear quantity ρ defined by the equation

$$J_z = M\rho^2 , \quad \textbf{(4.7)}$$

where M is the mass of the body.

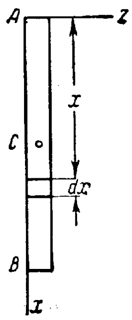
It follows from the definition that geometrically the radius of gyration is equal to the distance from the axis Oz to a point, such that if the mass of the whole body were concentrated in it the moment of inertia of the point would be equal to the moment of inertia of the whole body. Knowing the radius of gyration, we can obtain the moment of inertia of a body from Eq. (4.7) and vice versa.

4.4. Moments of Inertia of Some Homogeneous Bodies. If we divide a body into elements, in the limit the sum in Eq. (4.6) will become an integral and we obtain

$$J_z = \int\limits_{(V)} h^2 dm , \quad \textbf{(4.8)}$$

where the integration is over the whole volume of the body and depends on the coordinates of the points of the body. Eq. (4.8) is convenient in computing the moments of inertia of homogeneous bodies. Let us examine some examples.

1. Thin Homogeneous Rod of Length l and Mass M. Let us find its moment of inertia with respect to an axis Az perpendicular to the rod (Fig. 12). If we lay off a coordinate



axis Ax along $AB,$ for any line element of length dx we have $h = x$ and its mass $dm = \rho_1 dx,$ where $\rho_1 = M/l$ is the mass of a unit length of the rod, and Eq. (4.8) gives:

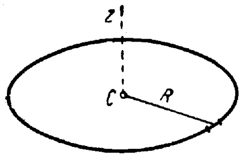
$$J_A = \int_0^l x^2 dm = \rho_1 \int_0^l x^2 dx = \rho_1 \frac{l^3}{3}.$$

Substituting the expression for ρ_1, we obtain finally

Fig.12.

$$J_A = \frac{1}{3} M l^2 .$$

2. Thin Circular Homogeneous Ring of Radius R and Mass M. Let us find its moment of inertia with respect to an axis Cz perpendicular to the plane of the ring through its centre (Fig. 13). As all the points of the ring are at a distance $h_k = R$ from axis $Cz,$ Eq. (4.6) gives



$$J_C = \sum m_k R^2 = (\sum m_k) R^2 = MR^2 .$$

Hence, for the ring

$$J_C = MR^2 .$$

Fig.13. It is evident that the same result is obtained for the moment of inertia of a cylindrical shell of mass M and radius R with respect to its axis.

3. Circular Homogeneous Disc or Cylinder of Radius R and Mass M. Let us compute the moment of inertia of a circular disc with respect to an axis Cz perpendicular to it through its centre (Fig. 14a). Consider an elemental ring of radius r and width dr. Its

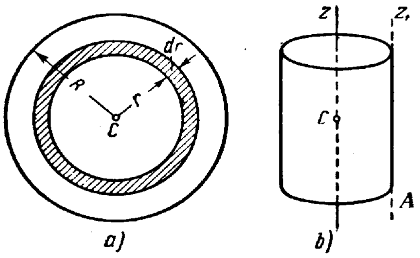


Fig.14.

area is $2\pi r dr$, and its mass $dm = \rho_2 2\pi r dr$, where $\rho_2 = \frac{M}{\pi R^2}$ is the mass of a unit area of the disc. From Eq.(4.8) we have for the elemental ring

$$dJ_C = r^2 dm = 2\pi\rho_2 r^3 dr$$

and for the whole disc

$$J_C = 2\pi\rho_2 \int_0^R r^3 dr = \frac{1}{2}\pi\rho_2 R^4 .$$

Substituting the expression for ρ_2 we obtain finally

$$J_C = \frac{1}{2} MR^2 .$$

It is evident that the same formula is obtained for the moment of inertia J_z of a homogeneous circular cylinder of mass M and radius R with respect to its axis Cz (Fig. 14b).

The moments of inertia of non-homogeneous and composite bodies can be determined experimentally with the help of appropriate instruments.

4.5. Moments of Inertia of a Body About Parallel Axes. The Parallel-Axis (Huygens') Theorem. In the most general case, the moments of inertia of the same body with respect to different axes are different. Let us see how to determine the moment of inertia of a body with respect to any axis if its moment of inertia with respect to a parallel axis through the body is known.

Draw an axis Cz through the centre of mass C of a body, and an axis Oz_1 parallel to it (Fig. 15), denoting the distance between the two axes by the symbol d. By definition we have

$$J_{Oz} = \sum m_k h_k^2 \,, J_{Cz} = \sum m_k h_k'^2 ,$$

where h_k is the distance of an arbitrary point B of the body from axis Oz_1, and h_k' is the distance of the same point from axis Cz, It follows from ΔBae that

$$h_k^2 = h_k'^2 + d^2 - 2dh_k' \cos\alpha_k .$$

Let us draw from point C, as the origin of a coordinate system, axes x and y perpendicular to Cz, such that x intersects with axis Oz_1.It is evident that $Cx||ae$. Denoting the coordinates of point B as x_k, y_k, z_k we obtain:

$$h_k^{'} \cos\alpha_k = x_k \text{ and } h_k^2 = h_k^{'2} + d^2 - 2dx_k \ .$$

Substituting this expression of h_k into the expression for J_{Oz_1} and taking the common factors d^2 and $2d$ outside the summation signs, we have

$$J_{Oz_1} = \sum m_k h_k^{'2} + \left(\sum m_k\right)d^2 - 2d\sum m_k x_k \ .$$

The first summation in the right side of the equation is equal to J_{Cz} and the second to the mass M of the body. Let us find the value of the third summation. From Eq. (4.4) we know that, for the coordinates of the centre of mass, $\sum m_k x_k = Mx_C$. But since in our case point C is the origin, $x_c = 0$, and consequently $\sum m_k x_k = 0$. We finally obtain

$$J_{Oz_1} = J_{Cz} + Md^2 \ . \qquad (4.9)$$

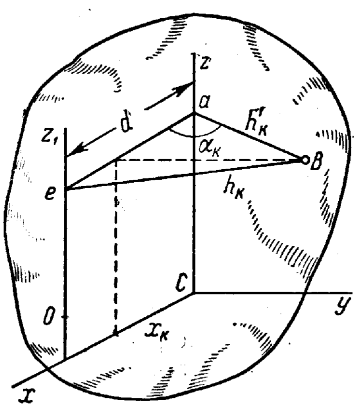


Fig.15.

Eq. (4.9) expresses the parallel-axis theorem enunciated by Huygens: *The moment of inertia of a body with respect to any axis is equal to the moment of inertia of the body with respect to a parallel axis through the centre of mass of the body plus the product of the mass of the body and the square of the distance between the two axes.*

It follows from Eq. (4.9) that $J_{Oz_1} > J_{Cz}$.Consequently, of all the axes of same direction, the moment of inertia is least with respect to the one through the centre of mass.

4.6. The Differential Equations of Motion of a System. Suppose we have a system of n particles. Choosing any particle of mass m_k belonging to the system, let us denote the resultant of all the external forces acting on the particle (both active forces and the forces of reaction) by the symbol $\vec{F}_k^e$ and the resultant of all the internal forces by $\vec{F}_k^i$. If the particle has an acceleration $\vec{w}_k$, then, by the fundamental law of dynamics,

$$m_k \vec{w}_k = \vec{F}_k^e + \vec{F}_k^i.$$

Similar results are obtained for any other particle, whence, for the whole system, we have

$$\left.\begin{aligned} m_1\vec{w}_1 &= \vec{F}_1^e + \vec{F}_1^i \\ m_2\vec{w}_2 &= \vec{F}_2^e + \vec{F}_2^i \\ &\dots\dots\dots\dots\dots\dots \\ m_n\vec{w}_n &= \vec{F}_n^e + \vec{F}_n^i \end{aligned}\right\} \qquad (4.10)$$

These equations, from which we can develop the law of motion of any particle of the system, are called *the differential equations of motion of a system* in vector form. Eqs.

(4.10) are differential because $\vec{w}_k = \frac{d\vec{v}_k}{dt} = \frac{d^2\vec{r}_k}{dt^2}$. In the most general case the forces in the right side of the equations depend on the time, the coordinates of the particles of the system, and their velocities.

By projecting Eqs. (4.10) on coordinate axes, we can obtain the differential equations of motion of a given system in terms of the projections on these axes.

The complete solution of the principal problem of dynamics for a system would be to develop the equation of motion for each particle of the system from the given forces by integrating the corresponding differential equations. For two reasons, however, this solution is not usually employed. Firstly, the solution is too involved and will almost inevitably lead into insurmountable mathematical difficulties. Secondly, in solving problems of mechanics it is usually sufficient to know certain overall characteristics of the motion of a system, without investigating the motion of each particle. These overall characteristics can be found with the help of the *general theorems* of systems dynamics, which we shall now study. The main application of Eqs.(4.10) or their corollaries will be to develop the respective general theorems.

5. GENERAL THEOREMS OF PARTICLE AND SYSTEM DYNAMICS

In solving many problems of dynamics it will be found that the so-called general theorems, representing corollaries of the fundamental law of dynamics, are more conveniently applied than the method of integration of differential equations of motion.

The importance of the general theorems is that they establish visual relationships between the principal dynamic characteristics of motion of material bodies, thereby presenting broad possibilities for analyzing the mechanical motions widely employed in practical engineering. Furthermore, the general theorems make it possible to study for practical purposes specific aspects of a given phenomenon without investigating the phenomenon as a whole. Finally, the use of the general theorems makes it unnecessary to carry out for every problem the operations of integration performed once and for all in proving the theorems, which simplifies the solution.

5.1. Momentum of a Particle and a System. One of the basic dynamic characteristics of particle motion is momentum (or linear momentum).

The momentum of a particle is defined as a vector quantity $m\vec{v}$ equal to the product of the mass of the particle and its velocity. The vector $m\vec{v}$ is directed in the same direction as the velocity, i.e., tangent to the path of the particle.

The linear momentum, or simply the momentum, of a system is defined as the vector quantity $\vec{Q}$ equal to the geometric sum (the principal vector) of the moments of all the particles of the system (Fig. 16):

$$\vec{Q} = \sum m_k \vec{v}_k \qquad (5.1)$$

It can be seen from the diagram that, irrespective of the velocities of the particles (provided that they are not parallel) the momentum vector can take any value, or even be zero when the polygon constructed with the vectors $m_k \vec{v}_k$ as its sides is closed. Consequently, the quantity $\vec{Q}$ does not characterize the motion of the system completely. Let us develop a formula with which it is much more convenient to compute $\vec{Q}$ and also to explain its meaning. It follows from Eq. (4.5) that

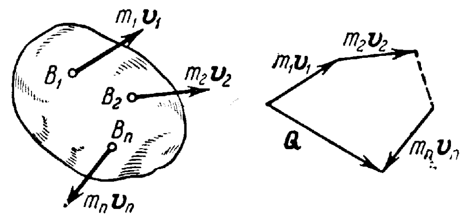


Fig.16.

$$\sum m_k \vec{r}_k = M\vec{r}_C.$$

Differentiating both sides with respect to time, we obtain

$$\sum m_k \frac{d\vec{r}_k}{dt} = M\frac{d\vec{r}_C}{dt} \text{ or } \sum m_k \vec{v}_k = M\vec{v}_C$$

whence we find that

$$\vec{Q} = M\vec{v}_c \qquad (5.2)$$

i.e., the momentum of a system is equal to the product of the mass of the whole system and the velocity of its center of mass. This equation is especially convenient in computing the momentum of rigid bodies.

It follows from Eq. (5.2) that, if the motion of a body (or a system) is such that the center of mass remains motionless, the momentum of the body is zero. Thus, the momentum of a body rotating about a fixed axis through its center of mass is zero (the polygon in Fig. 16 is closed).

If, on the other hand, a body has relative motion, the quantity $\vec{Q}$ will not characterize the rotational component of the motion about the center of mass. Thus, for a rolling wheel, $\vec{Q} = M\vec{v}_c$, regardless of how the wheel rotates about its center of mass *C*.

We see, therefore, that momentum characterizes only the translatory motion of a system, which is why it is often called linear momentum. In relative motion, the quantity $\vec{Q}$ characterizes only the translatory component of the motion of a system together with its center of mass.

5.2. Impulse of a Force. The concept of impulse (or linear impulse) of a force is used to characterize the effect on a body of a force acting during a certain interval of time. First let us introduce the concept of elementary impulse, i.e., impulse in an infinitesimal time interval *dt.* Elementary impulse is defined as a vector quantity $d\vec{S}$ equal to the product of the vector of the force $\vec{F}$ and the time element *dt:*

$$d\vec{S} = \vec{F}\, dt.$$

The elementary impulse is directed along the action line of the force.

The impulse $\vec{S}$ of any force $\vec{F}$ during a finite time interval t is computed as the integral sum of the respective elementary impulses:

$$\vec{S} = \int_0^{t_1} \vec{F} dt. \tag{5.3}$$

Thus, the impulse of a force in any time interval t_1 is equal to the integral of the elementary impulse over the interval from zero to t_1.

In the special case when the force $\vec{F}$ is of constant magnitude and direction ($\vec{F}$ = *const.),* we have $\vec{S} = \vec{F}\, t_1$, In the general case the magnitude of an impulse can be computed from its projections. We can find the projections of an impulse on a set of coordinate axes if we remember that an integral is the limit of a sum, and the projection of a vector sum on an axis is equal to the sum of the projections of the component vectors on the same axis. Hence,

$$S_x = \int_0^{t_1} F_x dt, \quad S_y = \int_0^{t_1} F_y dt, \quad S_z = \int_0^{t_1} F_z dt.$$

With these projections we can construct the vector $\vec{S}$ and find its magnitude and the angles it makes with the coordinate axes. The dimension of linear impulse in the technical system of units is [S]=kg-sec.

To solve the principal problem of dynamics, it is important to establish the forces whose impulses can be computed without knowing the equation of motion of the particle moving under the action of those forces. It is apparent that to these forces belong only constant forces and forces depending on time.

5.3. Theorem of the Motion of Center of Mass. In many cases the nature of the motion of a system (especially of a rigid body) is completely described by the law of motion of its center of mass. To develop this law, let us take the equations of motion of a system (4.10) and add separately their left and right sides. We obtain

$$\sum m_k \vec{w}_k = \sum \vec{F}_k^e + \sum \vec{F}_k^i . \tag{5.4}$$

Let us transform the left side of the equation. For the radius vector of the center of mass we have

$$\sum m_k \vec{r}_k = M\vec{r}_C .$$

Taking the second derivative of both sides of this equation with respect to time, and noting that the derivative of a sum equals the sum of the derivatives, we find

$$\sum m_k \frac{d^2\vec{r}_k}{dt^2} = M\frac{d^2\vec{r}_k}{dt^2}$$

or

$$\sum m_k \vec{w}_k = M\vec{w}_C$$

where $\vec{w}_c$ is the acceleration of the center of mass of the system. As the internal forces of a system give $\sum \vec{F}_k^i = 0$, by substituting all the developed expressions into Eq. (5.4), we obtain finally:

$$M\vec{w}_C = \sum \vec{F}_k^e . \qquad (5.5)$$

Eq. (5.5) states the theorem of the motion of the center of mass of a system. Its form coincides with that of the equation of motion of a particle of mass $m=M$ where the acting forces are equal to $\vec{F}_k^e$. We can therefore formulate the theorem of the motion of the centre of mass as follows: *the center of mass of a system moves as if it were a particle of mass equal to the mass of the whole system to which are applied all the external forces acting on the system.* Projecting both sides of Eq. (5.5) on the coordinate axes, we obtain

$$M\frac{d^2x_C}{dt^2} = \sum F_{kx}^e ,\ M\frac{d^2y_C}{dt^2} = \sum F_{ky}^e ,\ M\frac{d^2z_C}{dt^2} = \sum F_{kz}^e . \qquad (5.6)$$

These are the differential equations of motion of the center of mass in terms of the projections on the coordinate axes. The theorem is valuable for the following reasons:
1) It justifies the use of the methods of particle dynamics. It follows from Eqs. (5.6) that the solutions developed on the assumption that a given body is equivalent to a particle define the law of motion of the center of mass of that body. Thus, these solutions have a concrete meaning.
In particular, if a body is being translated, its motion is completely specified by the motion of its center of mass, and consequently, a body in translatory motion can always be treated as a particle of mass equal to the mass of the body. In all other cases, a body can be treated as a particle only when the position of its center of mass is sufficient to specify the position of the body.
2) The theorem makes it possible, in developing the equation of motion for the centre of mass of any system, to ignore all unknown internal forces. This is of special practical value.

5.4. The Law of Conservation of Motion of Center of Mass. The following important corollaries arise from the theorem of the motion of center of mass:
1) Let the sum of the external forces acting on a system be zero:

$$\sum \vec{F}_k^e = 0.$$

It follows, then, from Eq. (5.5) that $\vec{w}_c = 0$ or $\vec{v}_c = const.$ Thus, *if the sum of all the external forces acting on a system is zero, the center of mass of that system moves with a velocity of constant magnitude and direction, i.e., uniformly and rectilinearly.* In particular, if the center of mass was initially at rest it will remain at rest. The action of the internal forces, we see, does not affect the motion of the center of mass.
2) Let the sum of the external forces acting on a system be other than zero, but let the sum of their projections on one of the coordinate axes (the *x*- axis, for instance), be zero:

$$\sum F_{kx}^e = 0.$$

The first of Eqs. (5.6), then, gives

$$\frac{d^2x_C}{dt^2} = 0 \text{ or } \frac{dx_C}{dt} = \upsilon_{Cx} = const.$$

Thus, *if the sum of the projections on an axis of all the external forces acting on a system is zero, the projection of the velocity of the center of mass of the system on that axis is a constant quantity.* In particular, if at the initial moment v_{Cx}=0, it will remain zero at any subsequent instant, i.e., the center of mass of the system will not move along the x-axis (x_C = const.).

The above results express the law of conservation of motion of the center of mass of a system.

5.5. Theorem of the Change in the Momentum of a Particle. As the mass of a particle is constant, and its acceleration $\vec{w} = \frac{d\vec{v}}{dt}$ equation, which expresses the fundamental law of dynamics, can be expressed in the form:

$$\frac{d(m\vec{v})}{dt} = \sum \vec{F}_k \quad . \tag{5.7}$$

Let a particle of mass m moving under the action of a force $\vec{R} = \sum \vec{F}_k$ have a velocity $\vec{v}_0$ at time t=0, and at time t_1 let its velocity be $\vec{v}_1$ **.** Now let us multiply both sides of Eq. (5.7) by dt and take definite integrals. On the right side, where we integrate with respect to time, the limits of the integrals are zero and t_1; on the left side, where we integrate the velocity, the limits of the integral are the respective values of $\vec{v}_0$ and $\vec{v}_1$. As the integral of $d\,(m\vec{v}\,)$ is $m\vec{v}$, we have

$$m\vec{v}_1 - m\vec{v}_0 = \sum \int_0^{t_1} \vec{F}_k dt \, .$$

By Eq. (5.3), the integrals on the right side are the impulses of the acting force. Hence, we finally have

$$m\vec{v}_1 - m\vec{v}_0 = \sum \vec{S}_k . \tag{5.8}$$

Eq. (5.8) states the theorem of the change in the linear momentum of a particle: the change in the momentum of a particle during any time interval is equal to the geometric sum of the impulses of all the forces acting on the particle during that interval of time.

In problem solutions, projection equations are often used instead of the vector equation (5.8). Projecting both sides of Eq. (5.8) on a set of coordinate axes, we have

$$\left.\begin{aligned} mv_{1x} - mv_{0x} &= \sum S_{kx} \\ mv_{1y} - mv_{0y} &= \sum S_{ky} \\ mv_{1z} - mv_{0z} &= \sum S_{kz} \end{aligned}\right\} .$$

In the case of rectilinear motion along the x- axis, the theorem is stated by the first of these equations.

5.6. Theorem of the Change in Linear Momentum of the System. Consider a system of n particles. Writing the differential equations of motion (4.10) for this system and adding them, we obtain

$$\sum m_k \vec{w}_k = \sum \vec{F}_k^e + \sum \vec{F}_k^i .$$

From the property of internal forces the last summation is zero. Furthermore,

$$\sum m_k \vec{w}_k = \frac{d}{dt}\left(\sum m_k \vec{v}_k\right) = \frac{d\vec{Q}}{dt}$$

and we finally have

$$\frac{d\vec{Q}}{dt} = \sum \vec{F}_k^e \; . \tag{5.9}$$

Eq. (5.9) states the theorem of the change in the linear momentum of a system in differential form: *the derivative of the linear momentum of a system with respect to time is equal to the geometrical sum of all the external forces acting on the system.* In terms of projections on cartesian axes we have

$$\frac{dQ_x}{dt} = \sum F_{kx}^e \, , \frac{dQ_y}{dt} = \sum F_{ky}^e \, , \frac{dQ_z}{dt} = \sum F_{kz}^e \; . \tag{5.10}$$

Let us develop another expression for the theorem. Let the momentum of a system be $\vec{Q}_0$ at time $t=0,$ and at time t_1 let it be $\vec{Q}_1$. Multiplying both sides of Eq. (5.9) by dt and integrating, we obtain

$$\vec{Q}_1 - \vec{Q}_0 = \sum \int_0^{t_1} \vec{F}_k^e dt$$

or

$$\vec{Q}_1 - \vec{Q}_0 = \sum \vec{S}_k^e \tag{5.11}$$

as the integrals to the right give the impulses of the external forces. Eq. (5.11) states the theorem of the change in the linear momentum of a system in integral form: *the change in the linear momentum of a system during any time interval is equal to the sum of the impulses of the external forces acting on the body during the same interval of time.* In terms of projections on cartesian axes we have

$$\left.\begin{aligned} Q_{1x} - Q_{0x} &= \sum S_{kx}^e \\ Q_{1y} - Q_{0y} &= \sum S_{ky}^e \\ Q_{1z} - Q_{0z} &= \sum S_{kz}^e \end{aligned}\right\} .$$

Let us show the connection between this theorem and the theorem of the motion of center of mass. As $\vec{Q} = M\vec{v}_c$, by substituting this expression into Eq. (5.9) and taking into account that $\frac{d\vec{v}_c}{dt} = \vec{w}_c$ we obtain $M\vec{w}_c = \sum \vec{F}_k^e$ i.e., Eq. (5.5).

Consequently, the theorem of the motion of center of mass and the theorem of the change in the momentum of a system are, in effect, two forms of the same theorem. Whenever the motion of a rigid body (or system of bodies) is being investigated, both theorems may be used, though Eq. (5.5) is usually more convenient.

For a continuous medium (a fluid), however, the concept of center of mass of the whole system is virtually meaningless, and the theorem of the change in the momentum of a system is used in the solution of such problems.

The practical value of the theorem is that it enables us to exclude from consideration the immediately unknown internal forces (for instance, the reciprocal forces acting between the particles of a liquid).

5.7. The Law of Conservation of Linear Momentum. The following important corollaries arise from the theorem of the change in the momentum of a system:

1) Let the sum of all the external forces acting on a system be zero:

$$\sum \vec{F}_k^e = 0.$$

It follows from Eq. (5.9) that in this case $\vec{Q}$=*const.* Thus, *if the sum of all the external forces acting on a system is zero, the momentum vector of the system is constant in magnitude and direction.*

2) Let the external forces acting on a system be such that the sum of their projections on any axis *Ox* is zero:

$$\sum F_{kx}^e = 0.$$

It follows from Eqs. (5.10) that in this case $\vec{Q}_x$=*const.* Thus, *if the sum of the projections on any axis of all the external forces acting on a system is zero, the projection of the momentum of that system on that axis is a constant quantity.*

These results express the law of conservation of the linear momentum of a system.

5.8. Theorem of the Change in the Angular Momentum of a Particle. Often, in analyzing the motion of a particle, it is necessary to consider the change not of the vector $m\vec{v}$ itself, but of its moment. The moment of the vector $m\vec{v}$ with respect to any center *0* or axis *z* is denoted by the symbol $\vec{m}_0(m\vec{v})$ or $m_z(m\vec{v})$ and is called the moment of momentum or angular momentum with respect to that center or axis. The moment of vector $m\vec{v}$ is calculated in the same way as the moment of a force. Vector $m\vec{v}$ is considered to be applied to the moving particle. In magnitude $|\vec{m}_0(m\vec{v})| = mvh$, where h is the perpendicular distance from *0* to the position line of the vector $m\vec{v}$ (see Fig. 17).

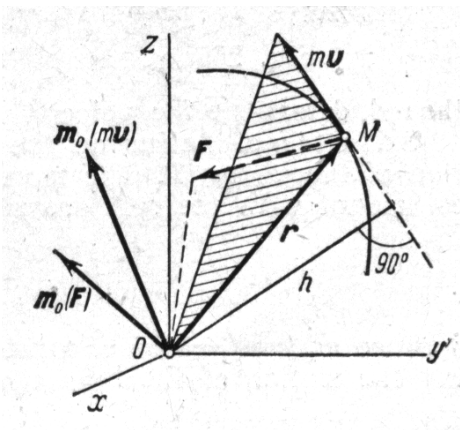


Fig.17.

1. *Principle of Moments About an Axis.* Consider a particle of mass *m* moving under the action of a force $\vec{F}$. Let us establish the dependence between the moments of the vectors $m\vec{v}$ and $\vec{F}$ with respect to any fixed axis *z*.

It is well known that

$$m_z(\vec{F}) = xF_y - yF_x. \qquad (5.12)$$

Similarly, for $m_z(m\vec{v})$, and taking m out of the parentheses, we have

$$m_z(m\vec{v}) = m(xv_y - yv_x)\,. \qquad (5.13)$$

Differentiating both sides of this equation with respect to time, we obtain

$$\frac{d}{dt}[m_z(m\vec{v})] = m\left(\frac{dx}{dt}v_y - \frac{dy}{dt}v_x\right) + \left(xm\frac{dv_y}{dt} - ym\frac{dv_x}{dt}\right).$$

The expression in the first parentheses of the right side of the equation is zero, as $\frac{dx}{dt} = v_x$ and $\frac{dy}{dt} = v_y$. From Eq. (5.12), the expression in the second pair of parentheses is equal to $m_z(\vec{F})$, since, from the fundamental law of dynamics,

$$m\frac{dv_x}{dt} = F_x\,,\ \ m\frac{dv_y}{dt} = F_{y.}$$

Finally, we have

$$\frac{d}{dt}[m_z(m\vec{v})] = m_z(\vec{F})\,. \qquad (5.14)$$

This equation states the principle of moments about an axis: *the derivative of the angular momentum of a particle about any axis with respect to time is equal to the moment of the acting force about the same axis*

2. *Principle of Moments about a Center.* Let us find for a particle moving under the action of a force $\vec{F}$ (Fig. 17) the relation between the moments of vectors $m\vec{v}$ and $\vec{F}$ with respect to any fixed center *0*. It was shown that $\vec{m}_0(\vec{F}) = \vec{r} \times \vec{F}$. Similarly, $\vec{m}_0(m\vec{v}) = \vec{r} \times m\vec{v}$.

Vector $\vec{m}_0(\vec{F})$ is normal to the plane through *0* and vector $\vec{F}$, while the vector $\vec{m}_0(m\vec{v})$ is normal to the plane through the center *0* and vector $m\vec{v}$. Differentiating the expression $\vec{m}_0(m\vec{v})$ with respect to time, we obtain

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = (\frac{d\vec{r}}{dt} \times m\vec{v}) + (\vec{r} \times m\frac{d\vec{v}}{dt}) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{w})\,.$$

But $\vec{v} \times m\vec{v} = 0$, as the vector product of two parallel vectors, and $m\vec{w} = \vec{F}$. Hence,

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r} \times \vec{F}\ \text{, or}\ \ \frac{d}{dt}[\vec{m}_0(m\vec{v})] = \vec{m}_0(\vec{F})\,. \qquad (5.15)$$

This is the principle of moments about a center: *the derivative of the angular momentum of a particle about any fixed center with respect to time is equal to the moment of the force acting on the particle about the same center.* An analogous theorem is true for the moments of vector $m\vec{v}$ and force $\vec{F}$ with respect to any axis *z*, which is evident if we project both sides of Eq. (5.15) on that axis. This was proved directly in item 1.

5.9. Total Angular Momentum of a System. *The total angular momentum of a system with respect to any center 0 is defined as the quantity* $\vec{K}_0$, *equal to the geometrical sum of the angular moments of all the particles of the system with respect to that center:*

$$\vec{K}_0 = \sum \vec{m}_0(m_k \vec{v}_k). \tag{5.16}$$

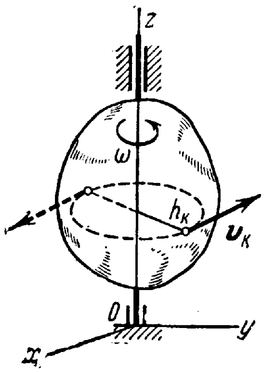


Fig.18.

The angular moment of a system with respect to each of three rectangular coordinate axes are found similarly:

$$K_x = \sum m_x(m_k \vec{v}_k),\ K_y = \sum m_y(m_k \vec{v}_k),\ K_z = \sum m_z(m_k \vec{v}_k). \tag{5.17}$$

By the theorem proved in § 5.8, K_x, K_y, K_z are the respective projections of vector $\vec{K}_0$ on the coordinate axes.

To understand the physical meaning of $\vec{K}$, let us compute the angular momentum of a rotating body with respect to its axis of rotation. If a body rotates about a fixed axis Oz (Fig. 18), the linear velocity of any particle of the body at a distance from the axis is ωh_k. Consequently, for that particle $m_z(m_k \vec{v}_k) = m_k v_k h_k = m_k \omega\, h_k^2$. Then, taking the common multiplier ω outside of the parentheses, we obtain for the whole body

$$K_z = \sum m_z(m_k \vec{v}_k) = (\sum m_k h_k^2)\omega.$$

The quantity in the parentheses is the moment of inertia of the body with respect to the z- axis (§ 4.3). We finally obtain

$$K_z = J_z\, \omega. \tag{5.18}$$

Thus, *the angular momentum of a rotating body with respect to the axis of rotation is equal to the product of the moment of inertia of the body and its angular velocity.*

If a system consists of several bodies rotating about the same axis, then, apparently,

$$K_z = J_{1z}\omega_1 + J_{2z}\omega_2 + \ldots + J_{nz}\omega_n.$$

The analogy between Eqs. (5.2) and (5.18) will be readily noticed: the momentum of a body is the product of its mass (the quantity characterizing the body's inertia in translatory motion) and its velocity; the angular momentum of a body is equal to the product of its moment of inertia (the quantity characterizing a body's inertia in rotational motion) and its angular velocity.

Just as the momentum of a system is a characteristic of its translatory motion, the total angular momentum of a system is a characteristic of its rotational motion.

5.10. Theorem of the Change in the Total Angular Momentum of a System. The principle of moments, which was proved for a single particle (§ 5.8), is valid for all the particles of a system. If, therefore, we consider a particle of mass m_k and velocity $\vec{v}_k$ belonging to a system, we have for that particle

$$\frac{d}{dt}\left[\vec{m}_0(m_k \vec{v}_k)\right] = \vec{m}_0(\vec{F}_k^e) + \vec{m}_0(\vec{F}_k^i).$$

where $\vec{F}^e{}_k$ and $\vec{F}^i{}_k$ are the resultants of all the external and internal forces acting on the particle.

Writing such equations for all the particles of the system and adding them, we obtain

$$\frac{d}{dt}\left[\sum \vec{m}_0(m_k\vec{v}_k)\right] = \sum \vec{m}_0(\vec{F}_k^e) + \sum \vec{m}_0(\vec{F}_k^i).$$

But from the properties of the internal forces of a system, the last summation vanishes. Hence, taking into account Eq. (5.16), we obtain finally

$$\frac{d\vec{K}_0}{dt} \sum \vec{m}_0(\vec{F}_k^e). \tag{5.19}$$

This equation states the following principle of moments for a system: *The derivative of the total angular momentum of a system about any fixed center with respect to time is equal to the sum of the moments of all the external forces acting on that system about that center.*

Projecting both sides of Eq. (5.19) on a set of fixed axes $Oxyz$, we obtain

$$\frac{dK_x}{dt} = \sum m_x(\vec{F}_k^e), \quad \frac{dK_y}{dt} = \sum m_y(\vec{F}_k^e), \quad \frac{dK_z}{dt} = \sum m_z(\vec{F}_k^e). \tag{5.20}$$

5.11. The Law of Conservation of the Total Angular Momentum. The following important corollaries can be derived from the principle of moments.

1) Let the sum of the moments of all the external forces acting on a system with respect to a center *0* be zero:

$$\sum \vec{m}_0(\vec{F}_k^e) = 0.$$

It follows, then, from Eq. (5.19) that $\vec{K}_0$ = *const.* Thus, *if the sum of the moments of all external forces acting on a system taken with respect to any center is zero, the total angular momentum of the system with respect to that center is constant in magnitude and direction.*

2) Let the external forces acting on a system be such that the sum of their moments with respect to any fixed axis Oz is zero:

$$\sum m_z(\vec{F}_k^e) = 0.$$

It follows, then, from Eqs. (5.20) that K_z = *const.* Thus, *if the sum of the moments of all the external forces acting on a system with respect to any axis is zero, the total angular momentum of the system with respect to that axis is constant.*

These conclusions express the law of conservation of the total angular momentum of a system. It follows from them that internal forces cannot change the total angular momentum of a system.

5.12. Kinetic Energy of Particle and a System. The kinetic energy of a particle is a scalar quantity equal $\frac{1}{2}mv^2$. *The kinetic energy of a system is defined as a scalar* *quantity T equal to the arithmetical sum of the kinetic energies of all the particles of the system:*

$$T = \sum \frac{m_k v^2{}_k}{2} \ . \tag{5.21}$$

If a system consists of several bodies, its kinetic energy is, evidently, equal to the sum of the kinetic energies of all the bodies:

$$T = \sum T_k \ .$$

Let us develop the equations for computing the kinetic energy of a body in different types of motion.

1. Translatory Motion. In this case all the points of a body have the same velocity, which is equal to the velocity of the centre of mass. Therefore, for any point $v_k = v_c$, and Eq. (5.21) gives

$$T_{trans} = \sum \frac{m_k v^2{}_k}{2} = \frac{1}{2}\left(\sum m_k\right) v_c^2 ,$$

$$T_{trans} = \frac{1}{2} M v_c^2 \ . \tag{5.22}$$

2. Rotational Motion. The velocity of any point of a body rotating about an axis *Oz* is $v_k = \omega h_k$ where h_k is the distance of the point from the axis of rotation, and ω is the angular velocity of the body. Substituting this expression into Eq. (5.21) and taking the common multipliers outside the parentheses we obtain

$$T_{rotation} = \sum \frac{m_k \omega^2 h^2{}_k}{2} = \frac{1}{2}\left(\sum m_k h_k^2\right)\omega^2 \ .$$

The term in the parentheses is the moment of inertia of the body with respect to the axis *z*. Thus we finally obtain

$$T_{rotation} = \frac{1}{2} J_z \omega^2 \ . \tag{5.23}$$

3. Plane Motion. In plane motion, the velocities of all the points of a body are at any instant directed as if the body were rotating about an axis perpendicular to the plane of motion and passing through the instantaneous centre of zero velocity *P* (Fig. 19). Hence, by Eq. (5.23)

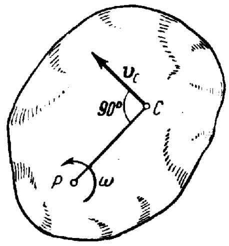


Fig.19.

$$T_{plane} = \frac{1}{2} J_p \omega^2$$

where J_p is the moment of inertia of the body with respect to the instantaneous axis of rotation.

The quantity J_p is variable, as the position of the centre *P* continuously changes with the motion of the body. Let us introduce instead of J_p a constant moment of inertia J_c with respect to an axis through the centre of mass *C* of the body. By the parallel-axis theorem, $J_p = J_c + Md^2$, where $d = PC$. Substituting this expression for J_p and taking into account that point P is the instantaneous centre of zero velocity and therefore ωd = $\omega PC = v_c$, where v_c is the velocity of the centre of mass, we obtain finally

$$T_{plane} = \frac{1}{2} M\ v_c^2 + \frac{1}{2} J_c \omega^2 . \qquad (5.24)$$

5.13. Work Done by a Force. Power. The concept of work is introduced as a measure of the action of a force on a body in a given displacement, specifically that action which is represented by the change in the magnitude of the velocity of a moving particle.

First let us introduce the concept of elementary work done by a force in an infinitesimal displacement *ds*. The elementary work done by a force $\vec{F}$ (Fig. 20) is defined as a scalar quantity

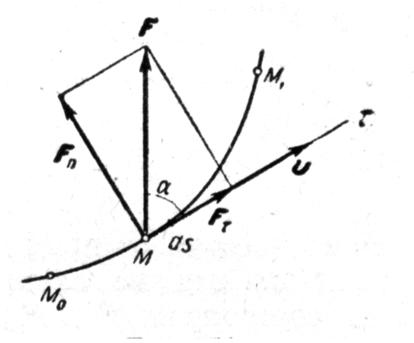


Fig.20.

$$dA = F_\tau ds , \qquad (5.25)$$

where F_τ is the projection of the force on the tangent to the path in the direction of the displacement, and *ds* is an infinitesimal displacement of the particle along that tangent.

This definition corresponds to the concept of work as a characteristic of that action of a force which tends to change the magnitude of velocity. For if force $\vec{F}$ is resolved into components $\vec{F}_\tau$ and $\vec{F}_n$, only the component $\vec{F}_\tau$, which imparts the particle its tangential acceleration, will change the magnitude of the velocity. Noting that $F_\tau = F\cos\alpha$, we further obtain from Eq. (5.25)

$$dA = F ds \cos\alpha . \qquad (5.26)$$

If angle α is acute, the work is of positive sense. In particular, at α=0, the elementary work *dA=Fds.*

If angle a is obtuse, the work is of negative sense. In particular, at α=180°, the elementary work *dA=—Fds.*

If angle α=90°, i.e., if a force is directed perpendicular to the displacement, the elementary work done by the force is zero.

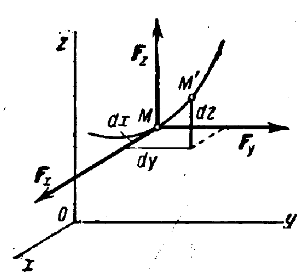


Fig.21.

Let us now find an analytical expression for elementary work. For this we resolve force F into components F_x, F_y, F_z, parallel to the coordinate axes (Fig. 21). The infinitesimal displacement *MM'=ds* is compounded of the displacements *dx, dy, dz* parallel to the coordinate axes, where *x, y, z* are the coordinates of point *M*. The work done by force *F* in the displacement *ds* can be calculated as the sum of the work done by its components F_x, F_y, F_z in the displacements *dx, dy, dz*. But the work in the displacement *dx* is done only by component F_x and is equal to $F_x dx$. The work in the displacements *dy* and *dz* is calculated similarly. Thus, we finally obtain

$$dA = F_x dx + F_y dy + F_z dz\,. \tag{5.27}$$

Eq. (5.27) gives the analytical expression of the elementary work done by a force.

The work done by a force in any finite displacement M_0M_1 (see Fig. 20) is calculated as the integral sum of the corresponding elementary works and is equal to

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} F_\tau ds\,. \tag{5.28}$$

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (F_x dx + F_y dy + F_z dz). \tag{5.29}$$

The limits of the integral correspond to the values of the variables of integration at points M_0 and M_1, (or, more exactly, the integral is taken along the curve M_1M_0 i.e., it is curvilinear).

In order to solve the principal problem of dynamics, it is important to establish the forces whose work can be calculated immediately without knowing the equation of motion of the particle on which they are acting. It can be seen that to these forces belong only constant forces or forces which depend on the position (coordinates) of a moving particle.

Without knowing the equation of motion of the particle, i. e., without first solving the principal problem of dynamics, the work done by such forces cannot be determined.

Power. The term *power* is defined as the work done by a force in a unit of time (the time rate of doing work). If work is done at a constant rate, the power

$$W = \frac{A}{t_1}$$

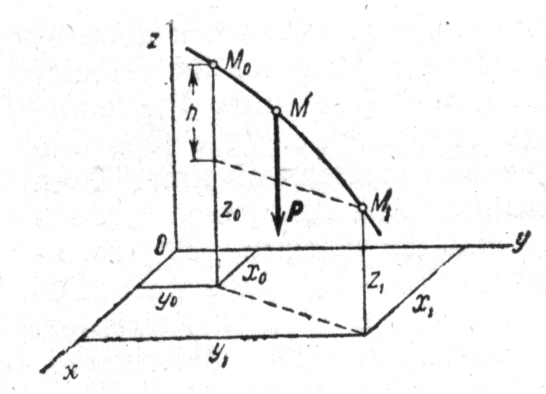
where t_1 is the time in which the work A is done. In the general case,

$$W = \frac{dA}{dt} = \frac{F_\tau ds}{dt} = F_\tau v\,.$$

It can be seen from the equation $W = F_\tau v$ that if a motor has a given power W, the tractive force F_τ is inversely proportional to the velocity v. That is why, for instance, on an upgrade or poor road a motor car goes into lower gear, thereby reducing the speed and developing a greater tractive force with the same power.

5.14. Examples of Calculation of Work. The examples considered below give results which can be used immediately in solving problems.

1) Work Done by a Gravitational Force. Let a particle M subjected to a gravitational force $\vec{P}$ be moving from a point $M_0(x_0,y_0,z_0)$ to a point $M_1(x_1,y_1,z_1)$. Choose a coordinate system so that the axis Oz would point vertically up (Fig. 22). Then $P_x=0$, $P_y=0$, $P_z=-P$. Substituting these expressions into Eq. (5.29) and taking into account that the integration variable is z, we obtain



$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-Pdz) = -P\int_{z_0}^{z_1} dz = P(z_0 - z_1)\,.$$

If point M_0 is higher than M_1 then $z_0 — z_1=h$, where h is the vertical displacement of the particle; if, on the other hand, M_0, is below M_1 then $z_0—z_1=—(z_1—z_0)=—h$. Finally we have

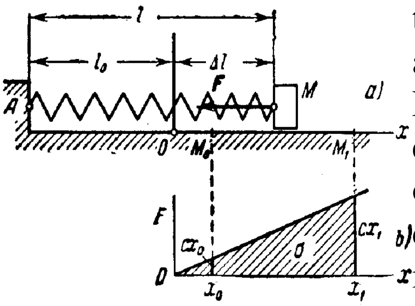
$$A_{(M_0M_1)} = \pm Ph\,.$$

Fig.22.

The work is positive if the initial point is higher than the final one and negative if it is lower.

It follows from this that the work done by gravity does not depend on the path along which the point of its application moves. Forces possessing this property are called conservative forces.

2) Work Done by an Elastic Force. Consider a weight M lying in a horizontal plane and attached to the free end of a spring (Fig. 23). Let point O on the plane represent the position of the end of the spring when it is not in tension ($AO = l_0$ is the length of the unextended spring) and let it be the origin of our coordinate system.



Now if we draw the weight from its position of equilibrium 0, stretching the spring to length l, acting on the weight will be the elastic force of the spring F directed towards 0. According to Hooke's Law, the magnitude of this force is proportional to the extension of the spring Δl

Fig.23.

$=l—l_0$. As in our case $\Delta l=x$, then in magnitude $F = c|\Delta l| = c|x|$.

The factor c is called the stiffness of the spring, or the spring constant. Let us find the work done by the elastic force in the displacement of the weight from position $M_0(x_0)$ to position $M_1(x_1)$. As in this case $F_x=—F=—cx$, $F_y=F_z=0$, then, substituting these expressions into Eq. (5.29), we obtain

$$A_{(M_1M_0)} = \int_{(M_0)}^{(M_1)} (-cx)dx = -c\int_{x_0}^{x_1} xdx = \frac{c}{2}(x_0^{\,2} - x_1^{\,2})\,.$$

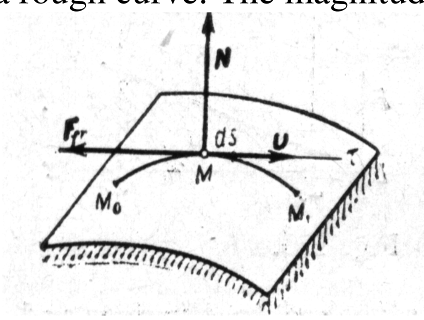
In the obtained formula x_0 is the initial extension of the spring Δl , and x_1 is the final extension Δl_{fin}. Hence

$$A_{(M_0M_1)} = \frac{c}{2}[(\Delta l_{in})^2 - (\Delta l_{fin})^2]\,.$$

The work is positive if $|\Delta l_{in}| \rangle |\Delta l_{fin}|$, i.e., when the end of the spring moves towards the position of equilibrium, and negative when, $|\Delta l_{in}| \langle |\Delta l_{fin}|$ i.e., when the end of the spring moves away from the position of equilibrium.

It follows, therefore, that the work done by the force F depends only on the quantities Δl_{in} and Δl_{fin} and does not depend on the actual path traveled by M. Consequently,an elastic force is also a conservative force.

3) Work Done by Friction. Consider a particle moving on a rough surface (Fig. 24) or a rough curve. The magnitude of the frictional force acting on the particle is *fN,* where f is the coefficient of friction and N is the normal reaction of the surface. Frictional force is directed opposite to the displacement of the particle, whence $F_{fr\tau} = —fN$, and from Eq. (5.28),



$$A_{(M_0M_1)} = -\int_{(M_0)}^{(M_1)} F_{fr} ds = -\int_{(M_0)}^{(M_1)} fN ds.$$

Fig.24.

If the frictional force is constant, then, $A_{(M_0M_1)} = -F_{fr}s$ where s is the length of the arc M_0M_1 along which the particle moves. Thus, the work done by kinetic friction is always negative. It depends on the length of the arc M_0M_1 and consequently it is non-conservative.

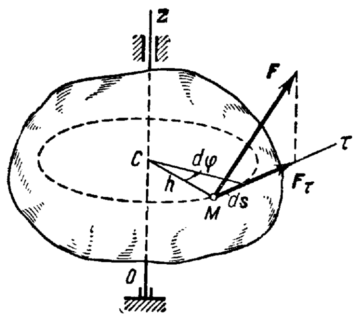
4) Work Done by Gravitational Forces Acting on a System. The work done by a gravitational force acting on a particle of weight p_k will be $p_k(z_{ko} — z_{k1})$ where z_{ko} and z_{k1} are the coordinates of the initial and final positions of the particle . Then the total work done by all the gravitational forces acting on a system will be

$$A = \sum p_k z_{k0} - \sum p_k z_{k1} = P(z_{C0} - z_{C1}) = \pm Ph_C$$

where P is the weight of the system, and h_C is the vertical displacement of the centre of gravity (or centre of mass) of the system.

5) Work Done by Forces Applied to a Rotating Body. The elemental work done by the force $\vec{F}$ applied to the body in Fig. 25 will be

$$dA = F_\tau ds = F_\tau h d\varphi$$



since $ds = hd\varphi$, where $d\varphi$ is the angle of rotation of the body.

But it is evident that $F_\tau h = m_z(\vec{F})$. We shall call the quantity $M_z = m_z(\vec{F})$ the *turning moment,* or *torque.* Thus we obtain

$$dA = M_z d\varphi. \qquad (5.30)$$

Fig.25.

Eq. (5.30) is valid when several forces are acting, if it is assumed that $M_z = \sum m_z(\vec{F}_k)$. The work done in a turn through a finite angle φ will be

$$A = \int_0^{\varphi 1} Mz d\varphi. \qquad (5.31)$$

and, for a constant torque (M_z = const.),

$$A = M_z \varphi_1 \,. \tag{5.32}$$

If acting on a body is a force couple lying in a plane normal to Oz, then, evidently, M_z in Eqs. (5.30)-(5.32) will denote the moment of that couple.

Let us see how power is determined in this case. From Eq. (5.30) we find

$$M = \frac{dA}{dt} = \frac{M_z d\varphi}{dt} = M_z \omega \,.$$

6) Work Done by Frictional Forces Acting on a Rolling Body. A wheel of radius R rolling without slipping on a plane (surface) is subjected to the action of a frictional force $\vec{F}_{fr}$, which prevents the slipping of the point of contact B on the surface.

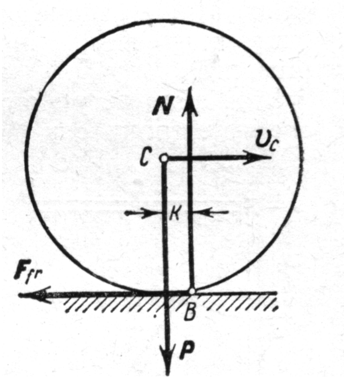


Fig.26.

The elemental work done by this force is $dA = -F_{fr} ds_B$. But point B is the instantaneous centre of velocity , and $\vec{v}_B = 0$. As $ds_B = v_B dt$, $ds_B = 0$, and for every elemental displacement $dA=0$.

Thus, in rolling without slipping, the work done by the frictional force preventing slipping is zero in any displacement of the body. For the same reason, the work done by the normal reaction N is also zero.

The resistance to rolling is created by the couple $(\vec{N}, \vec{P})$ of moment $M = kN$, where k is the coefficient of rolling friction . Then by Eq. (5.30), and taking into account that the angle of rotation of a rolling wheel is $d\varphi = \frac{ds_C}{R}$

$$dA_{roll} = -kNd\varphi = -\frac{k}{R} N ds_C \tag{5.33}$$

where ds_C is the elemental displacement of the centre C of the wheel. If N = *const.*, then the total work done by the forces resisting rolling will be

$$A_{roll} = -kN\varphi_1 = -\frac{k}{R} N s_C \,. \tag{5.34}$$

5.15. Theorem of the Change in the Kinetic Energy of a Particle. Consider a particle of mass m displaced by acting forces from a position M_0 where its velocity is $\vec{v}_0$, to position M_1 where its velocity is $\vec{v}_1$.

To obtain the required relation, consider the equation $m\vec{w} = \sum \vec{F}_k$, which expresses the fundamental law of dynamics. Projecting both parts of this equation on the tangent M_τ to the path of the particle in the direction of motion, we obtain $m\, w_\tau = \Sigma F_{k\tau}$.

The tangential acceleration in the left side of the equation can be written in the form $w_\tau = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds} v$, hence we have $mv\frac{dv}{ds} = \sum F_{k\tau}$.

Multiplying both sides of the equation by ds, bring m under the differential sign.

Then, noting that $F_{k\tau}ds=dA_k$, where dA_k is the elementary work done by the force $\vec{F}_k$, we obtain an expression of the theorem of the change in kinetic energy in differential form:

$$d\left(\frac{mv^2}{2}\right)=\sum dA_k \,. \tag{5.35}$$

Integrating both sides of Eq. (5.35) in the limits between the corresponding values of the variables at points M_0 and M_1, we finally obtain

$$\frac{mv_1^{\;2}}{2}-\frac{mv_0^{\;2}}{2}=\sum_k A_k \,. \tag{5.36}$$

Eq. (5.36) states the theorem of the change in the kinetic energy of a particle in the final form: *the change in the kinetic energy of a particle in any displacement is equal to the algebraic sum of the work done by all the forces' acting on the particle in the same displacement.*

The Case of Constrained Motion. If the motion of a particle is constrained, then, from, the left side of Eq. (5.36) will include the work done by the given (active) forces $\vec{F}_k^{\;a}$ and the work done by the reaction force of the constraint. Let us limit ourselves to the case of a particle moving on a fixed smooth (frictionless) surface or curve. In this case the reaction N is normal to the path of the particle, and N_τ=0. Then by Eq. (5.28), the work done by the reaction force of a fixed smooth surface (or curve) in any displacement of a particle is zero, and from Eq. (5.36) we obtain

$$\frac{mv_1^2}{2}-\frac{mv_0^2}{2}=\sum A^a{}_{(M_0M_1)}.$$

Thus, in a displacement of a particle on a fixed smooth surface (or curve) the change in the kinetic energy of the particle is equal to the sum of the work done in this displacement by the active forces applied to that particle.

If the surface, (curve) is not smooth, the work done by frictional force will be added to the work done by the active forces.

5.16. Theorem of the Change in the Kinetic Energy of a System. The theorem proved in § 5.15 is valid for any point of a system. Therefore, if we take any particle of mass m_k and velocity $\vec{v}_k$ belonging to a system, we have for this particle

$$\frac{m_k v_{k1}^2}{2}-\frac{m_k v_{k0}^2}{2}=A_k^e+A_k^i$$

where $\vec{v}_{k0}$ and $\vec{v}_{k1}$ denote the particle's velocity at the beginning and the end of the displacement, and A_k^e and A_k^i are the sums of the work done by all the external and internal forces acting on the particle through this displacement.

If we write similar equations for all the particles of a system and add them up, we obtain

$$\sum \frac{m_k \upsilon_{k1}^2}{2} - \sum \frac{m_k \upsilon_{k0}^2}{2} = \sum A_k^e + \sum A_k^i \qquad \text{or}$$

$$T_1 - T_0 = \sum A_k{}^e + \sum A_k{}^i \qquad (5.37)$$

where T_1 and T_0 denote the kinetic energy of the system at the beginning and the end of the displacement.

This equation states the following theorem of the change in kinetic energy: *The change in the kinetic energy of a system during any displacement is equal to the sum of the work done by all the external and internal forces acting on the system in that displacement.*

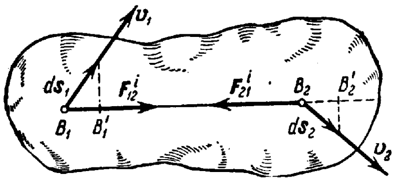


Fig.27.

For an infinitesimal displacement of a the system theorem takes the form

$$dT = dA^e - dA^i \qquad (5.38)$$

where dA^e and dA^i denote the elemental work done by all the external and internal forces acting on the system.

Unlike the previously proved theorems, in Eqs. (5.37) and (5.38) the internal forces are not ignored. For, if $\vec{F}_{12}^i$ and $\vec{F}_{21}^i$ are the forces of interaction between points B_1 and B_2, of a system (see Fig. 27), then $\vec{F}_{12}^i + \vec{F}_{21}^i = 0$, but at the same time point B_1 may be moving towards B_2 and point B_2, towards B_1. The work done by each force is positive, and the total work will not be zero.

The Case of Non-Deformable Systems. A non-deformable system is defined as one in which the distance between the points of application of the internal forces does not change during the motion of the system. Special cases of such systems are a rigid body and an inextensible string. Let two points B_1 and B_2 of a non-deformable system (Fig. 27) be acting on each other with forces $\vec{F}_{12}^i$ and $\vec{F}_{21}^i$ ($\vec{F}_{12}^i = -\vec{F}_{21}^i$) and let their velocities at some instant be $\vec{v}_1$ and $\vec{v}_2$ Their displacements in a time interval dt will be $ds_1 = \upsilon_1 dt$ and $ds_2 = \upsilon_2 dt$ directed along vectors $\vec{v}_1$ and $\vec{v}_2$. But as line B_1B_2 is non-deformable, it follows from the laws of kinematics that the projections of vectors $\vec{v}_1$ and $\vec{v}_2$ and consequently of the displacements ds_1 and ds_2 on the direction of B_1B_2 will be equal, i.e., $B_1B_1 = B_2B_2$. Then the elemental work done by forces $\vec{F}_{12}^i$ and $\vec{F}_{21}^i$ will be equal in magnitude and opposite in sense, and their sum will be zero. This holds for all internal forces in any displacement of a system.

We conclude from this that the sum of the work done by all the internal forces of a non-deformable system is zero, and Eq. (5.37) takes the form

$$T_1 - T_0 = \sum A_k^e. \qquad (5.39)$$

Both the external and internal forces in Eqs. (5.37)-(5.39) include the reactions of constraints. If the constraints on which the bodies of a system move are smooth, then the work done by the reactions of these constraints in any displacement of the system is zero and the reactions will not enter into Eqs (5.37)-(5.39).

Thus in applying the theorem of the change in kinetic energy to frictionless systems, all the immediately unknown reactions of the constraints will be excluded from the problem. This is where its practical value lies.

6. THE PRINCIPLES OF DYNAMICS

6.1. D'Alembert's Principle for a Particle and a System. Consider a particle M moving along a given fixed curve or surface (Fig. 28). The resultant of all the active forces applied to the particle is denoted by the symbol $\vec{F}^a$. If the action of the constraint is replaced by its reaction $\vec{N}$, the particle can be considered as a free one moving under the action of forces $\vec{F}^a$ and $\vec{N}$. Let us see what force $\vec{F}^i$ should be added to the forces $\vec{F}^a$ and $\vec{N}$ to balance them. If the resultant of the forces $\vec{F}^a$ and $\vec{N}$ is $\vec{R}$, then, obviously, the required force $\vec{F}^i = -\vec{R}$.

Let us express force $\vec{F}^i$ in terms of the acceleration of the moving particle. As, according to the fundamental law of dynamics, $\vec{R} = m\vec{w}$, $\vec{F}^i = -m\vec{w}$.

The force $\vec{F}^i$, equal in magnitude to the product of the mass of the particle and its acceleration and directed oppositely to the acceleration, is called the *inertia force* of the particle *M.*

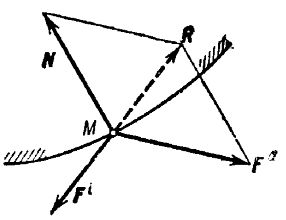


Fig.28.

Thus, if to the forces $\vec{F}^a$ and $\vec{N}$ is added the inertia force $\vec{F}^i$, the forces will be balanced, and we will have

$$\vec{F}^a + \vec{N} + \vec{F}^i = 0. \qquad (6.1)$$

This equation states *D'Alembert's principle for a particle: if at any given moment to the active forces and the reactions of the constraints acting on a particle is added the inertia force, the resultant force system will be in equilibrium and all the equations of statics will apply to it.*

D'Alembert's principle provides a method of solving problems of dynamics by developing equations of motion in the form of equations of equilibrium.

In applying D'Alembert's principle it should always be remembered that actually only forces $\vec{F}^a$ and $\vec{N}$ are acting on a particle and that the particle is in motion. The inertia force does not act on a moving particle and the concept is introduced for the sole purpose of developing equations of dynamics with the help of the simpler methods of statics.

D'Alembert's Principle for a System. Consider a system of n particles. Let us select any particle of mass m_k and denote the resultants of all the external and internal forces applied to it by the symbols $\vec{F}_k^{ext}$ and $\vec{F}_k^{\mathrm{int}}$. If we add to these forces the inertia force $\vec{F}_k^i = -m_k\vec{w}_k$ then according to D'Alembert's principle for a single particle the force system $\vec{F}_k^{ext}, \vec{F}_k^{\mathrm{int}}, \vec{F}_k^i$ will be in equilibrium, and consequently,

$$\vec{F}_k^{ext} + \vec{F}_k^{\mathrm{int}} + \vec{F}_k^i = 0.$$

Reasoning similarly for all the particles of the system, we arrive at the following result, which expresses D'Alembert's principle for a system: *If at any moment of time* *to the effective external and internal forces acting on every particle of a system are added the respective inertia forces, the resultant force system will be in equilibrium and all the equations of statics will apply to it.*

We know from statics that the geometrical sum of balanced forces and the sum of their moments with respect to any centre *0* are zero; we know, further, from the principle of solidification, that this holds good not only for forces acting on a rigid body, but for any deformable system. Thus, according to D'Alembert's principle, we must have

$$\sum\left(\vec{F}_k^{ext} + \vec{F}_k^{\text{int}} + \vec{F}_k^{\,i}\right),$$
$$\sum\left[\vec{m}_0(\vec{F}_k^{ext}) + \vec{m}_0(\vec{F}_k^{\text{int}}) + \vec{m}_0(\vec{F}_k^{i})\right].$$

Let us introduce the following notation:

$$\vec{R}^i = \sum \vec{F}_k^i\,,\ \vec{M}_0^i = \sum \vec{m}_0(\vec{F}_k^i)\,.$$

The quantities $\vec{R}^i$ and $\vec{M}_0^i$ are respectively the principal vector of the inertia forces and their principal moment with respect to a centre *0.* Taking into account that the sum of the internal forces and the sum of their moments are each zero we obtain

$$\sum \vec{F}_k^{ext} + \vec{R}^i = 0\,,\ \sum \vec{m}_0(\vec{F}_k^{ext}) + \vec{M}_0^i = 0\,. \qquad (6.2)$$

Use of Eqs. (6.2), which follow from D'Alembert's principle, simplifies the process of problem solution because the equations do not contain the internal forces. Actually Eqs. (6.2) are equivalent to the equations expressing the theorems of the change in the momentum and the total angular momentum of a system, differing from them only in form.

6.2. The Principal Vector and the Principal Moment of the Inertia Forces of a Rigid Body. It follows from the Statics that a system of inertia forces applied to a rigid body can be replaced by a single force equal to $\vec{R}^i$ and applied at the centre *0*, and a couple of moment $\vec{M}_0^i$.The principal vector of a system, it will be recalled, does not depend on the centre of reduction and can be computed at once. As $\vec{F}_k^i = -m_k\vec{w}_k$ then taking into account §5.3, we will have:

$$\vec{R}^i = -\sum m_k\vec{w}_k = -M\vec{w}_c\,. \qquad (6.3)$$

Thus, *the principal vector of the inertia forces of a moving body is equal to the product of the mass of the body and the acceleration of its centre of mass, and is opposite in direction to the acceleration.*

Let us determine the principal moment of the inertia forces for particular types of motion.

1. Translatory Motion. In this case a body has no rotation about its centre of mass *C,* from which we conclude that $\sum \vec{m}_0(\vec{F}_k^{ext}) = 0$, and Eq. (6.2) gives $\vec{M}_0^i = 0$.

Thus, *in translatory motion, the inertia forces of a rigid body can be reduced to a single resultant* $\vec{R}^i$ *through the centre of mass of the body.*

2. Plane Motion. Let a body have a plane of symmetry, and
let it be moving parallel to the plane. By virtue of symmetry, the principal vector and

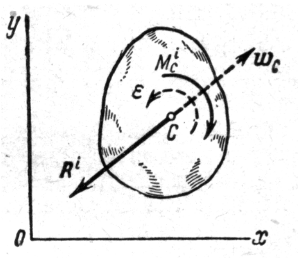


Fig.29.

the resultant couple of inertia forces lie, together with the centre of mass *C,* in that plane.

Therefore, placing the centre of reduction in point *C,* we obtain from Eq. (6.2) $M_C^i = -\sum m_C(\vec{F}_k^{ext})$. On the other hand (see § 5.9,5.10), $\sum m_C(\vec{F}_k^{ext}) = J_C\varepsilon$. We conclude from this that

$$M_C^i = -J_C\varepsilon. \qquad (6.4)$$

Thus, in such motion *a system of inertia forces can be reduced to a resultant force* $\vec{R}^i$ *[Eq. (6.3)I applied at the centre of mass C (Fig. 29) and a couple in the plane of symmetry of the body whose moment is given by Eq. (6.4).* The minus sign shows that the moment M_c^i is in the opposite direction of the angular acceleration of the body.

3. Rotation about an Axis through the Centre of Mass. Let a body have a plane of symmetry, and let the axis of rotation C_z be normal to the plane through the centre of mass. This case will thus be a particular case of the previous motion. But here $\vec{w}_c$ *=0,* and consequently, $\vec{R}^i$ *=0.*

Thus, in this case *a system of inertia forces can be reduced to a couple in the plane of symmetry of the body of moment*

$$M_z^i = -J_z\varepsilon.$$

In applying Eqs. (6.3) and (6.4) to problem solutions, the magnitudes of the respective quantities are computed and the directions are shown in a diagram.

6.3. Virtual Displacements of a System. Degrees of Freedom. In determining the equilibrium conditions of a system by the methods of so-called graphical statics we had to consider the equilibrium of every body separately, replacing the action of all applied constraints by the unknown reaction forces. When the number of bodies in a system is large, this method becomes cumbersome, involving the solution of a large number of equations with many unknown quantities.

Now we shall make use of a number of kinematical and dynamical concepts to investigate a more general method for the solution of problems of statics, which makes it possible to determine at once, the equilibrium conditions for any mechanical system. The basic difference between this method and the methods of geometrical statics is that the action of constraints is taken into account not by introducing the reaction forces but by investigating the possible displacements of a system if its equilibrium were disturbed. These displacements are known in mechanics by the name of *virtual displacements.*

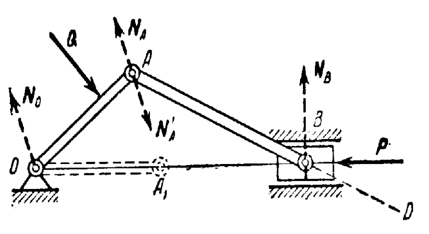


Fig.30.

Virtual displacements of the particles of a system must satisfy two conditions: 1) they must be infinitesimal, since if a displacement is finite the system will occupy a new configuration in which the equilibrium conditions may be different; 2) they must be consistent with the constraints of the system, as otherwise we should change the character of the mechanical system under consideration. For instance, in the crankshaft mechanism in Fig. 30, a displacement of the points of the crank *OA* into configuration OA_1 cannot considered as a virtual displacement, as the equilibrium conditions under the action of forces *P* and *Q* will be have changed. At the same time, even an infinitesimal displacement of point *B* of the connecting rod along *BD* would not be a virtual displacement: it would have been possible if the slides at *B* were replaced by a rocker, i.e., if it were a different mechanism.

Thus, we shall define as a virtual displacement of a system the sum total of any arbitrary infinitesimal displacements of the particles of the system consistent with all the constraints acting on the system at the given instant. We shall denote the virtual displacement of any point by an elementary vector $\delta\vec{s}$ in the direction of the displacement.

In the most general case, the particles and bodies of a system may have a number of different virtual displacements (not considering $\delta\vec{s}$ and $-\delta\vec{s}$ as being different). For every system, however, depending on the type of constraints, we can specify a certain number of independent virtual displacements such that any other virtual displacements will be obtained as their geometrical sum. For example, a bead lying on a horizontal plane can move in many directions on the plane. Nevertheless, any virtual displacement $\delta\vec{s}$ may be produced as the sum of two displacements $\delta\vec{s}_1$ and $\delta\vec{s}_2$ along two mutual perpendicular horizontal axes ($\delta\vec{s} = \delta\vec{s}_1 + \delta\vec{s}_2$).

The number of possible mutually independent displacements of a system is called the number of degrees of freedom of that system. Thus, a bead on a plane (regarded as a particle) has two degrees of freedom. A crankshaft mechanism, evidently, has one degree of freedom. A free particle has three degrees of freedom (three independent displacements along mutually perpendicular axes). A free rigid body has six degrees of freedom (three translatory displacements along orthogonal axes and three rotations about those axes).

Ideal Constraints. If a particle has for a constraint a smooth surface the reaction $\vec{N}$ of the constraint is normal to the surface and the elementary work done by the force $\vec{N}$ in any virtual displacement of the particle is zero. It was shown that if we neglect rolling friction, the sum of the work done by the reaction forces $\vec{N}$ and $\vec{F}_{fr}$ in any virtual displacement of a rolling body is also zero. The internal forces of any non-deformable system also possess this property.

Let us introduce the following notation: the elementary work done by an active force $\vec{F}^a$ in any virtual displacement $\delta\vec{s}$ — the virtual work—shall be denoted by the symbol A^a ($\delta A^a = F^a\, \delta s \cos\alpha$, where α is the angle between the directions of the force and the displacement), and the virtual work done by the reaction $\vec{N}$ of a constraint, by the symbol δA^N. Then for all the constraints considered here,

$$\sum \delta A_k^N = 0. \tag{6.5}$$

Constraints, in which the sum of the virtual work produced by all the reaction forces in any virtual displacement of a system is zero, are called *ideal constraints.*

We have seen that to such constraints belong all frictionless constraints along which a body slides and all rough constraints when a body rolls along them, neglecting rolling friction.

6.4. The Principle of Virtual Work. Consider a system of material particles in equilibrium under the action of the applied forces and constraints, assuming all the constraints imposed on the system to be ideal. Let us take an arbitrary particle B_k belonging to the system and denote the resultant of all the applied active forces (both external and internal) by the symbol $\vec{F}_k^a$, and the resultant of all the reactions of the constraints (also external and internal) by the symbol $\vec{N}_k$. Then, since point B_k is in equilibrium together with the system, $\vec{F}_k^a + \vec{N}_k = 0$ or $\vec{N}_k = -\vec{F}_k^a$.

Consequently, in any virtual displacement of point B_k the virtual work δA_k^a and δA_k^N done by the forces $\vec{F}_k^a$ and $\vec{N}_k$ are equal in magnitude and opposite in sense and therefore vanish, i.e., we have:

$$\delta A_k^a + \delta A_k^N = 0.$$

Reasoning in the same way we obtain similar equations for all the particles of a system, adding which we obtain

$$\Sigma\delta A_k^a + \Sigma\delta A_k^N = 0.$$

But from the property of ideal constraints (6.5), the second summation is zero, whence

$$\Sigma\delta A_k^a = 0, \tag{6.6}$$

or

$$\sum (F_k^a \delta s_k \cos\alpha_k) = 0. \tag{6.7}$$

We have thus proved that if a mechanical system with ideal constraints is in equilibrium, the active forces applied to it satisfy the condition (6.6). The reverse is also true, i.e., if the active forces satisfy the condition (6.6), the system is in equilibrium. From this follows the principle of virtual work: *the necessary and sufficient conditions for the equilibrium of a system subjected to ideal constraints is that the total virtual work done by all the active forces is equal to zero for any and all virtual displacements consistent with the constraints.* Mathematically the necessary and sufficient condition for the equilibrium of any mechanical system is expressed by Eq. (6.6).

In analytical form this condition can be expressed as follows:

$$\sum (F_{kx}^a \delta x_k + F_{ky}^a \delta y_k + F_{kz}^a \delta z_k) = 0. \tag{6.8}$$

In Eq. (6.8) δx_k, δy_k, δz_k are the projections of the virtual displacements $\delta\vec{s}_k$ of point B_k on the coordinate axes. They are equal to the infinitesimal increments to the position coordinates of the point in its displacement and are computed in the same way as the differentials of coordinates.

The principle of virtual work provides in general form the equilibrium conditions of any mechanical system, whereas the methods of geometrical statics require the consideration of the equilibrium of every body of the system separately. Furthermore, application of the principle of virtual work requires that only the active forces be considered and makes it possible to ignore all the unknown reactions of constraints, when the constraints are ideal.

6.5. The General Equation of Dynamics. The principle of virtual work gives a general method for solving problems of statics. On the other hand, D'Alembert's principle makes it possible to employ the methods of statics in solving dynamical problems. It seems obvious that by combining both these principles we can develop a general method for the solution of problems of dynamics.

Consider a system of material particles subjected to ideal constraints. If we add to all the particles subjected to active forces $\vec{F}_k^a$ and the reaction forces $\vec{N}_k$ the corresponding inertia forces $\vec{F}_k^i = -m_k \vec{w}_k$, then by D'Alembert's principle the resulting force system will be in equilibrium. If we now apply the principle of virtual work, we obtain

$$\sum \delta A_k^a + \sum \delta A_k^i + \sum \delta A_k^N = 0 .$$

But from Eq. (6.5) the last summation is zero, and we finally obtain

$$\sum \delta A_k^a + \sum \delta A_k^i = 0 . \qquad (6.9)$$

Equation (6.9) represents the general equation of dynamics. It states that *in a moving system with ideal constraints the total virtual work done by all the active forces and all the inertia forces in any virtual displacement is zero at any instant.*

In analytical form Eq. (6.9) gives

$$\sum \left[\left(F_{kx}^a + F_{kx}^i\right)\delta x_k + \left(F_{ky}^a + F_{ky}^i\right)\delta y_k + \left(F_{kz}^a + F_{kz}^i\right)\delta z_k \right]. \qquad (6.10)$$

Equation (6.9) and (6.10) make it possible to develop the equations of motion for any mechanical system.

If a system consists of a number of rigid bodies, the relevant equations can be developed if to the active forces applied to each body are added a force equal to the principal vector of the inertia forces applied at any center, and a couple of moment equal to the principal moment of the inertia forces with respect to that center. Then the principle of virtual work can be used.

Упорядник

Долгов Олександр Михайлович

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